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Abstract

Recently we have introduced Moran type interacting particle systems in order to solve numerically normalized continuous time Feynman-Kac formulae. These schemes can also be seen as approximating procedures for certain simple generalized spatially homogeneous Boltzmann equations, so strong propagation of chaos is known to hold for them. We will give a new proof of this result by studying the evolution of tensorized empirical measures and then applying two straightforward coupling arguments. The only difficulty is in the first step to find nice martingales, and this will be done via the introduction of another family of Moran semigroups. This work also procures us the opportuneness to present an appropriate abstract setting, in particular without any topological assumption on the state space, and to apply a genealogical algorithm for the smoothing problem in nonlinear filtering context.

Keywords: Feynman-Kac formulae, canonical progressive processes, perturbations of general Markov processes by jump bounded generators, interacting particle systems, weak and strong propagation of chaos, tensorized empirical measures, Moran semigroups and martingales, coupling, genealogical processes and smoothing problems in nonlinear filtering.

AMS codes: 60F17, 60K35, 60J25, 65C05, 65D30, 60J75, 60G44, 60F05 and 60G35.

1 Introduction

The purpose of this article is to present a new proof of the strong propagation of chaos for the Moran interacting particle systems approximating continuous time and general space Feynman-Kac formulae.

Without going into the full details of our (over extended) setting, which will be given in next section, let us recall the latter problem (our original motivation comes from the design of Monte-Carlo methods to solve numerically nonlinear filtering equations, see for instance the short explanation given at the beginning of section 5, but one could find extra applications in other fields, cf the discussion included in [6]): on a measurable state space $(E, \mathcal{E})$, we are provided with a progressive time inhomogeneous Markovian process $(X_t)_{t\geq 0}$ and with a bounded measurable function $U : \mathbb{R}_+ \times E \to \mathbb{R}_+$, and we want to estimate the normalized Feynman-Kac formulae

$$\eta_t(\varphi) \quad \stackrel{\text{def.}}{=} \quad \frac{\mathbb{E}\left[\varphi(X_t)\exp\left(\int_0^t U_s(X_s)\,ds\right)\right]}{\mathbb{E}\left[\exp\left(\int_0^t U_s(X_s)\,ds\right)\right]} \tag{1}$$

for any $t \geq 0$ and any bounded measurable function $\varphi : E \to \mathbb{R}$ (from now on, $\mathbb{E}$ will designate the expectation relative to any underlying probability $\mathbb{P}$).

To deal with this question, we proposed in [5] a scheme of interacting particle systems: for all $N \geq 1$ (which stands for the number of particles), we construct a Markovian process $\xi^{(N)} = (\xi_t^{(N)})_{t\geq 0} = ((\xi_t^{(N,1)}, \xi_t^{(N,2)}, \cdots, \xi_t^{(N,N)}))_{t\geq 0}$ taking values in E^N (see the section 2.4 below), such that in the limit of a large number of particles, for $t \geq 0$ given, the empirical measure

$$\eta_t^{(N)} \quad \stackrel{\text{def.}}{=} \quad \frac{1}{N}\sum_{1\leq i\leq N} \delta_{\xi_t^{(N,i)}} \tag{2}$$

is quite close to the probability η_t defined by the equations (1).

In the basic principle of its mechanism, the process $\xi^{(N)}$ looks like a Moran particle system (for a general description of this kind of genetic algorithms, see [3]), but the renormalization with respect to N that we have chosen for its selection part make it closer to a Nanbu interacting system (cf for instance [10]) associated to a simple generalized spatially homogeneous Boltzmann equation.

A standard problem in the related literature is the strong propagation of chaos, ie we are wondering if the law of a fixed particle, say $(\xi_t^{(N,1)})_{t\geq 0}$, converge in the total variation sense on any compact time interval toward the law of some natural time inhomogeneous Markovian process $\bar{X} = (\bar{X}_t)_{t\geq 0}$. The latter is often called the nonlinear process (or sometimes the tarjet process) associated to the underlying generalized spatially homogeneous Boltzmann equation, because its evolution at any given time also depends on the marginal law of the process at this instant.

This result could in fact be obtained via an application of the interacting graph approach developed in the works of Graham and Méléard ([12], [10]), one has just to make some arrangements to deal with the fact that our general definition of the process X (which in particular has to be time inhomogeneous for the applications we have in mind) and our unusual assumptions on its state space (where even no topological structure is assumed) are not exactly the same, but these appear as perhaps not really essential difficulties. Anyway, the next section describing precisely our set-up, the basic manipulations it allows and the precautions which have to be taken, is not completely useless in this respect, since it would have to be considered if one want to extend their state space (which was assumed to be $\mathbb{R}^n$, $n \in \mathbb{N}^*$). One motivation for this generalization (apart from its theoretical consistency) is that it enables one to treat path spaces as state spaces, and this has some practical applications.

Nevertheless, we prefer to introduce an other method, maybe more immediate (e.g. without any reference to Botzmann trees ...), because we will take into account that our models are simpler, since we are not in the need of considering the broader setting of [10]. More precisely, in our case, we have a nice a priori explicit expression (1) for the (deterministic) limiting objects, making them appear as ratios of linear terms with respect to η_0, which is hidden in $\mathbb{E}$ as the initial distribution. This structure is more tractable than the information one would get by merely looking at the nonlinear equation of evolution satisfied by the family $(\eta_t)_{t\geq 0}$ (in particular the problem of unicity for its solution(s) has not to be taken in account). So we can make use of some associated nonnegative Feynman-Kac semigroups to obtain without much difficulty the weak propagation of chaos for tensorized empirical measures (it corresponds to the convergence of the moments of the empirical measures, following the terminology of [3], except that we will have here to consider the empirical measures at different times). This will be done in section 3, following the spirit already prevailing in [5] and [6], for the usual (ie nontensorized) empirical measures.

Then two quite straightforward coupling arguments (respectively presented in subsections 4.1 and 4.2) enable us to conclude to the expected result. It should be noticed that the upper bounds we obtain for this convergence, are of the same order in the numbers of particles as those got by Graham and Méléard [10].

In the last and short section, we explain an application of our considerations to the genealogical process associated to the particle system which is related to the practical issue of smoothing in nonlinear filtering.

Finally we think that the study of the tensorized empirical measures is interesting in itself and could be developed further (by obtaining for instance related central limit theorems), it furthermore illustrate the flexibility of the semigroup approach and make clear some links with the general theory of measure valued process (cf. [3]).

2 The setting

As indicated before, we have try in this paper to work under very minimalist assumptions. This kind of considerations about an adequate axiomatization of a Markov processes theory (as close as possible to the general measurable Markov chains theory) may appear like an idle game, but we played it anyway, in order to fix a robust framework and to see which unnecessary structures (mainly the topological ones) can be removed from our previous works. To fulfill this side goal, our approach will have to differ in some aspects from the one presented in [6], but it will be adapted to the proof of the strong propagation of chaos, which in the end will (almost) always be satisfied. Along, we will show that even the weak condition considered in [6] was in fact useless to get the weak propagation of chaos. The principal difference is that the Markovian process X will be very general, and in particular we will make no explicit reference to the "carrés du champ".

At the present stage, we are still wondering if this set-up is sufficient to obtain the central limit theorem shown in [6], but this question will not be investigated here.

2.1 Hypotheses on the model

We begin by presenting the rigorous definition of the objects entering into the composition of the rhs of (1):

• The abstract state space E is merely endowed with a measurable structure, and $\mathcal{E}$ will denote its σ-algebra. We will also designate by $\mathcal{B}_{\mathrm{b}}(E)$ and $\mathcal{P}(E)$ respectively the set of all bounded measurable functions (equiped with the supremum norm $\|\cdot\|$) and the set of all probabilities on E.

• The measurable mapping

$$U : \mathbb{R}_+ \times E \ni (t,x) \mapsto U_t(x) \in \mathbb{R}_+$$

will only be supposed to be locally bounded in time, in the sense that for all $T \geq 0$, its restriction to $[0,T] \times E$ is bounded. We will then denote

$$u_T \stackrel{\text{def.}}{=} \sup_{0\leq t\leq T,\, x\in E} U_t(x) < +\infty$$

• The measurable process X appearing in (1) will be defined as a canonical coordinate process, under an appropriate "inhomogeneous" family of probabilities, ensuring that it satisfies the Markov property. So the first problem to be tackled is the definition of the space of "canonical trajectories":

A priori one would just consider $\mathbf{M}(\mathbb{R}_+, E)$ the set of all measurable paths from $\mathbb{R}_+$ to E, but as we will try to explain it later, this space is too large to be handled efficiently. Nevertheless $X = (X_t)_{t\geq 0}$ will designate the related process of canonical coordinates or its restriction to any of the subset of $\mathbf{M}(\mathbb{R}_+, E)$. Thus we make the assumption that we are given a nonempty set $\mathbb{M}(\mathbb{R}_+, E) \subset \mathbf{M}(\mathbb{R}_+, E)$ satisfying the condition (so in particular we also have $E \neq \emptyset$):

(H1) *Consists of the next two points:*

- *If we are given a sequence $(\omega_i)_{i\geq 0}$ of elements of $\mathbb{M}(\mathbb{R}_+, E)$ and a strictly increasing sequence $(t_i)_{i\geq 0}$ of nonnegative real numbers, satisfying $t_0 = 0$ and $\lim_{i\to\infty} t_i = +\infty$, then the element $\omega \in \mathbf{M}(\mathbb{R}_+, E)$ defined by*

$$\forall\, i \geq 0,\, \forall\, t_i \leq s < t_{i+1}, \qquad X_s(\omega) = X_s(\omega_i)$$

  belongs to $\mathbb{M}(\mathbb{R}_+, E)$.

- *Let $\mathcal{M}(\mathbb{R}_+, E)$ be the σ-field generated by the coordinates $(X_t)_{t\geq 0}$ on $\mathbb{M}(\mathbb{R}_+, E)$. Then the mapping*

$$\mathbb{R}_+ \times \mathbb{M}(\mathbb{R}_+, E) \ni (t,\omega) \mapsto X_t(\omega) \in E$$

  is $\mathcal{R}_+ \otimes \mathcal{M}(\mathbb{R}_+, E)$-measurable, where $\mathcal{R}_+$ denotes the usual Borelian σ-field on $\mathbb{R}_+$ (quite similarly, for any Borelian set $I \subset \mathbb{R}_+$, $\mathcal{R}_I$ will stand for the trace of $\mathcal{R}_+$ on I).

We will discuss about this condition (H1) in the remarks at the end of this section, but in the whole subsequent development, we will assume that a particular element $\diamond \in \mathbb{M}(\mathbb{R}_+, E)$ has been chosen (it will often play the role of a cemetery point).

For $t \geq 0$, let $\mathbb{M}([t,+\infty[, E) \subset \mathbf{M}([t,+\infty[, E)$ be the image of $\mathbb{M}(\mathbb{R}_+, E)$ under the mapping $(X_s)_{s\geq t}$; it is the set of all "admissible" paths after time t. We endow it naturally with the σ-field $\mathcal{M}([t,+\infty[, E)$ generated by the variables $\{X_s : s \geq t\}$. As usually, we will also have to consider on $\mathbb{M}([t,+\infty[, E)$ the filtration $(\mathcal{M}([t,s], E))_{s\geq t}$, where for any interval I of $\mathbb{R}_+$, $\mathcal{M}(I, E)$ will designate $\sigma(X_u\,;\, u \in I)$.

Note that for $0 \leq t \leq s$, the mapping

$$[t,s] \times \mathbb{M}([t,+\infty[, E) \ni (u,\omega) \mapsto X_u(\omega) \in E$$

is $\mathcal{R}_{[t,s]} \otimes \mathcal{M}([t,s],E)$-measurable.

Our main object is a given family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ of probabilities respectively defined on $(\mathbb{M}([t,+\infty[,E),\mathcal{M}([t,+\infty[,E))$ and satisfying:

- An initial condition parametrization property: for all $t \geq 0$ and $x \in E$, we have $X_t \circ \mathbb{P}_{t,x} = \delta_x$, the Dirac mass at x.

- A regularity property: for all $t \geq 0$ and $A \in \mathcal{M}([t,+\infty[,E)$, the mapping

$$E \ni x \quad \mapsto \quad \mathbb{P}_{t,x}[A]$$

is $\mathcal{E}$-measurable.

- A Markovian compatibility property: for all $0 \leq t \leq s$, all $x \in E$ and all $A \in \mathcal{M}(]s,+\infty[,E)$, we have $\mathbb{P}_{t,x}$-a.s. the following equality for the conditional expectation:

$$\mathbb{P}_{t,x}[A|\mathcal{M}([t,s],E)] \quad = \quad \mathbb{P}_{s,X_s}[A]$$

Taking into account the initial condition parametrization, it appears that this equality is in fact true for all $A \in \mathcal{M}([s,+\infty[,E)$, but reciprocally, note that this "extended" assumption does not imply the initial condition parametrization property.

From now on, such a family will be called Markovian.

Thus for all fixed $(t,x) \in \mathbb{R}_+ \times E$, the process $(X_s)_{s\geq t}$ is Markovian under $\mathbb{P}_{t,x}$. More generally, using the measurability assumption above, for any distribution $\eta_0 \in \mathcal{P}(E)$, we can define a probability $\mathbb{P}_{\eta_0}$ on $(\mathbb{M}([0,+\infty[,E),\mathcal{M}([0,+\infty[,E))$, by stating that

$$\forall\ A \in \mathcal{M}([0,+\infty[,E), \qquad \mathbb{P}_{\eta_0}[A] \quad = \quad \int_E \mathbb{P}_{0,x}[A]\, \eta_0(dx)$$

($\mathbb{E}_{\eta_0}$ will stand for the expectation relative to $\mathbb{P}_{\eta_0}$, the probability $\eta_0 \in \mathcal{P}(E)$ being fixed, and in (1) we should now replace $\mathbb{E}$ by $\mathbb{E}_{\eta_0}$). Then $X = (X_s)_{s\geq 0}$ is also easily seen to be Markovian under $\mathbb{P}_{\eta_0}$, and the distribution of X_0 is η_0.

As $t \geq 0$ varies, the probabilities $\mathbb{P}_{t,x}$, for $x \in E$, are defined on different measurable spaces, and this fact can be annoying for the formulation of some properties. So for any fixed $t \geq 0$, we introduce the injection

$$I_t \,:\, \mathbb{M}([t,+\infty[,E) \quad \to \quad \mathbb{M}(\mathbb{R}_+,E)$$

defined by

$$\forall\ s \geq 0,\, \forall\ \omega \in \mathbb{M}([t,+\infty[,E), \qquad X_s(I_t(\omega)) \quad = \quad \begin{cases} X_s(\omega_0) & \text{, for } s < t \\ X_s(\omega) & \text{, for } s \geq t \end{cases}$$

This mapping is clearly measurable, so it enables us to see $\mathbb{P}_{t,x}$ as a probability on $(\mathbb{M}(\mathbb{R}_+,E),\mathcal{M}(\mathbb{R}_+,E))$, for all $x \in E$, and we will keep abusing of the same notation (ie "$\mathbb{P}_{t,x} = I_T \circ \mathbb{P}_{t,x}$"). Then our second and principal hypothesis just says that the Markovian family has some "time regularity":

(H2) *For all $A \in \mathcal{M}(\mathbb{R}_+,E)$, the mapping*

$$\mathbb{R}_+ \times E \ni (t,x) \mapsto \mathbb{P}_{t,x}[A]$$

is $\mathcal{R}_+ \otimes \mathcal{E}$-measurable.

As a consequence of the monotonous class theorem, it appears that for all bounded measurable functions $f : \mathbb{R}_+ \times E \times \mathbb{M}(\mathbb{R}_+, E)$, the mapping

$$\mathbb{R}_+ \times E \ni (t,x) \mapsto \int f(t,x,\omega)\, \mathbb{P}_{t,x}(d\omega)$$

is measurable.

Let us now introduce some functions which will be very interesting in the subsequent developement:

For all fixed $T \geq 0$, $V \in \mathcal{B}_{\mathrm{b}}([0,T]\times E)$ and $\varphi \in \mathcal{B}_{\mathrm{b}}(E)$, we define the mapping

$$F_{T,V,\varphi} \,:\, [0,T]\times E \ni (t,x) \;\mapsto\; \mathbb{E}_{t,x}\left[\exp\left(\int_t^T V_s(X_s)\, ds\right)\varphi(X_T)\right] \tag{3}$$

The consideration of the assumptions (H1) and (H2) and the measurability part of the Fubini theorem enable us to see that $F_{T,V,\varphi}$ is indeed $\mathcal{R}_{[0,T]}\otimes\mathcal{E}$-measurable.

But this mapping has some more interesting properties, and to present them, let us associate to the family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ the following general time-space martingale problems: for every fixed $T \geq 0$, we denote by $\mathcal{A}_T$ the vector space of functions $f \in \mathcal{B}_{\mathrm{b}}([0,T]\times E)$ for which there exists another function $L_T(f) \in \mathcal{B}_{\mathrm{b}}([0,T]\times E)$ such that for every fixed $(t,x)\in[0,T]\times E$, the process $(M_{t,s}(f))_{t\leq s\leq T}$ defined by

$$\begin{aligned} \forall\, t \leq s \leq T, \qquad M_{t,s}(f) &= f(s,X_s) - f(t,X_t) - \int_t^s L_T(f)(u,X_u)\, du \\ (&= M_{0,s}(f) - M_{0,t}(f)\,) \end{aligned} \tag{4}$$

is a $(\mathcal{M}([t,s],E))_{t\leq s\leq T}$-martingale under $\mathbb{P}_{t,x}$.

In this article the martingales will not be implicitly supposed to be càdlàg (a.s.), because this is not relevent to our setting. More accurately, noncàdlàg martingales appear naturally in our calculations, even if we had made more restrictive assumptions (cf. [6]), and in fact these occurences contributed to our choice of an extended set-up. But when we will need elementary stochastic calculus, we will have to consider a càdlàg version of the martingales at hand, and each time we have carefully verified that one can carry out the classical modification via an extension of the filtration (eg see [8], note also that all our martingales will be bounded). An example of the kind of the manipulations we have to resort to will be developed in the proof of lemma 2.1 below.

Remark that maybe $L_T(f)$ is not uniquely determined by $f\in\mathcal{A}_T$ (but again we will keep abusing of these notations), nevertheless this is not really important, since for martingale problems, one can consider multivalued operators, cf [9].

Here is the only preliminary result we will need, and which is somewhat well known in the theory of Feynman-Kac formulae:

Lemma 2.1 *For all fixed $T>0$, $V \in \mathcal{B}_{\mathrm{b}}([0,T]\times E)$ and $\varphi\in\mathcal{B}_{\mathrm{b}}(E)$, the mapping $F_{T,V,\varphi}$ belongs to $\mathcal{A}_T$, and we can (and will) take*

$$\forall\, 0\leq t\leq T,\, \forall\, x\in E, \qquad L_T(F_{T,V,\varphi})(t,x) \;=\; -V_t(x)F_{T,V,\varphi}(t,x)$$

Proof:

We have already seen above that $F_{T,V,\varphi}\in\mathcal{B}_{\mathrm{b}}([0,T]\times E)$. Now let us denote for any fixed $T>0$, $V\in\mathcal{B}_{\mathrm{b}}([0,T]\times E)$ and $\varphi\in\mathcal{B}_{\mathrm{b}}(E)$,

$$\forall\, 0\leq t\leq T, \qquad N_t \;=\; F_{T,V,\varphi}(t,X_t)$$

we will show that $(N_t - N_0 + \int_0^t V_s(X_s)N_s\,ds)_{0\leq t\leq T}$ is a (a priori not necessarily càdlàg) martingale under $\mathbb{P}_{\eta_0}$, for any given $\eta_0 \in \mathcal{P}(E)$. The more general requirement (for all initial conditions $(t,x) \in [0,T]\times E \ldots$) is proved in the same way, and the announced results follow.

The Markov property of X implies that the process $(M_t)_{0\leq t\leq T}$ defined by

$$\begin{aligned}\forall\, 0\leq t\leq T, \qquad M_t &= \exp\left(\int_0^t V_s(X_s)\,ds\right) N_t \\ &= \mathbb{E}_{\eta_0}\left[\exp\left(\int_0^T V_s(X_s)\,ds\right)\varphi(X_T)\middle| \mathcal{M}([0,t],E)\right]\end{aligned}$$

is a martingale. As we have no information about its time regularity (except the measurability), we will go into all the details of the calculations, which otherwise would be immediate (just remove the subscripts $^+$ from (5)).

Let $\mathcal{N}$ be the set of all $\mathbb{P}_{\eta_0}$-negligeable subsets, we denote for $t\geq 0$,

$$\begin{aligned}\mathcal{M}_t^+ &= \mathcal{N} \vee \bigcap_{s>t} \mathcal{M}([0,s],E) \\ M_t^+ &= \limsup_{s\in\mathbb{Q}\cap]t,+\infty[,\, s\to t} M_s\end{aligned}$$

then it is well known (see for instance [8]) that $(M_t^+)_{t\geq 0}$ is a $(\mathcal{M}_t^+)_{t\geq 0}$ càdlàg martingale such that for all $t\geq 0$,

$$M_t = \mathbb{E}_{\eta_0}[M_t^+|\mathcal{M}([0,t],E)]$$

With some obvious notations, we have

$$\begin{aligned}N_t^+ &= \exp\left(-\int_0^t V_s(X_s)\,ds\right) M_t^+ \\ &= N_0 + \int_0^t \exp\left(-\int_0^s V_u(X_u)\,du\right)\,dM_s^+ - \int_0^t V_s(X_s)N_s^+\,ds \end{aligned} \tag{5}$$

So let $s\geq 0$ and $A\in\mathcal{M}([0,s],E)$ be given, from the previous equality we get that

$$\mathbb{E}\left[\left(N_t^+ - N_s^+ + \int_s^t V_u(X_u)N_u^+\,du\right)\mathbb{1}_A\right] = 0$$

nevertheless what we do want to show is

$$\mathbb{E}\left[\left(N_t - N_s + \int_s^t V_u(X_u)N_u\,du\right)\mathbb{1}_A\right] = 0$$

But it is quite clear that

$$\begin{aligned}\mathbb{E}[N_t^+\mathbb{1}_A] &= \mathbb{E}[N_t\mathbb{1}_A] \\ \mathbb{E}[N_s^+\mathbb{1}_A] &= \mathbb{E}[N_s\mathbb{1}_A]\end{aligned}$$

and furthermore, using the fact that the mapping

$$[s,t]\times\mathbb{M}(\mathbb{R}_+,E)\ni(u,\omega) \;\mapsto\; V_u(X_u(\omega))N_u(\omega)$$

is measurable and the Fubini's theorem, we obtain that

$$\begin{aligned}\mathbb{E}\left[\left(\int_s^t V_u(X_u)N_u^+\,du\right)\mathbf{1}_A\right] &= \int_s^t \mathbb{E}\left[V_u(X_u)N_u^+\mathbf{1}_A\right]du\\ &= \int_s^t \mathbb{E}\left[V_u(X_u)N_u\mathbf{1}_A\right]du\\ &= \mathbb{E}\left[\left(\int_s^t V_u(X_u)N_u\,du\right)\mathbf{1}_A\right]\end{aligned}$$

from which our above assertion follows.

■

Remarks 2.2:

a) As here we will mainly work with a finite horizon $T \geq 0$, ie we will only consider the restriction of the Markovian family to the path space $\mathbb{M}([0,T],E)$, we could have replaced the first point of (H1) by the simplest following one:

• *If* ω_0 *and* ω_1 *are elements of* $\mathbb{M}(\mathbb{R}_+,E)$ *and* $t>0$ *is given, then the element* $\omega\in\mathbf{M}(\mathbb{R}_+,E)$ *defined by*

$$\forall\ s\geq 0,\qquad X_s(\omega)\ =\ \begin{cases} X_s(\omega_0), & \textit{if } s<t\\ X_s(\omega_1), & \textit{if } s\geq t\end{cases}$$

belongs to $\mathbb{M}(\mathbb{R}_+,E)$.

Then by induction, the first point of (H1) is true, but for finite sequences of times, and that is the only thing we need on a bounded interval $[0,T]$.

b) The hypothesis (H1) can be seen as an ersatz for the lack of regularity of the trajectories, and from this point of view, its important condition is the second point, which corresponds to the traditional notion of progressive process. In fact, as soon as $\mathcal{E}$ is not a trivial σ-algebra, $\mathbf{M}(\mathbb{R}_+,E)$ does not satisfy (H1): just note that for any given $A\in\sigma(X_t\,;\,t\geq 0)$, there exist a sequence $(t_i)_{i\geq 0}$ of nonnegative real numbers and a measurable set $A'\in\mathcal{E}^{\otimes\mathbb{N}}$ such that

$$A\ =\ \{\omega\in\mathbf{M}(\mathbb{R}_+,E)\ :\ (X_{t_i}(\omega))_{i\geq 0}\in A'\}$$

But let $\varphi\in\mathcal{B}_{\mathrm{b}}(E)$ taking at least the two values 0 and 1. Then the previous caracterization shows that the set

$$\{\omega\in\mathbf{M}(\mathbb{R}_+,E)\ :\ \int_0^1\varphi(X_u(\omega))\,du\ >\ 0\}$$

cannot belong to $\sigma(X_t\,;\,t\geq 0)$, whereas it should if $\mathbf{M}(\mathbb{R}_+,E)$ was to verify (H1).

c) Nevertheless the hypothesis (H1) may appear somewhat strange at a first look. In order to try to persuade the reader that it is in fact natural, we will present the first approach we thought of and its drawback. So at the beginning we considered the whole set $\mathbf{M}(\mathbb{R}_+,E)$ and to get rid of measurability problems we believed it was sufficient to endow it with the σ-algebra $\mathcal{F}(\mathbb{R}_+,E)$ generated by the coordinates and by the mappings

$$\mathbf{M}(\mathbb{R}_+,E)\ni\omega\ \mapsto\ \int_0^t\varphi(X_s(\omega))\,ds$$

for all $t\geq 0$ and all $\varphi\in\mathcal{B}_{\mathrm{b}}(E)$.

For a while, everything worked nice, via extensive use of the monotonous class theorem, even the construction given in next section can be completed. Furthermore, in this context, the family of functions

$$(F_{T,V,\varphi})_{T\geq 0,\, V\in\mathcal{B}_{\mathrm{b}}([0,T]\times E),\, \varphi\in\mathcal{B}_{\mathrm{b}}(E)} \tag{6}$$

is quite a basic object, as it is easily seen to determine the family of probabilities $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ (ie (6) plays the usual role of the associated semigroup, see below). So if lemma 2.1 was still true, we could for instance deduce a unicity property for the abstract martingale problems (defined as above) associated to the Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$.

But the trouble arises with this lemma 2.1 (which is fundamental with respect to the properties of Feynman-Kac formulae); we were unable to show it because in this set-up we could not use the Fubini theorem at the end of the above proof !

In order to go round this difficulty, we could have try to extend further $\mathcal{F}(\mathbb{R}_+, E)$, but for instance we do not know if the second point of (H1) is verified even with $(\mathbf{M}(\mathbb{R}_+, E), \mathcal{P}(\mathbb{R}_+, E))$ instead of $(\mathbb{M}(\mathbb{R}_+, E), \mathcal{M}(\mathbb{R}_+, E))$, where $\mathcal{P}(\mathbb{R}_+, E)$ is the total σ-field of all the subsets of $\mathbf{M}(\mathbb{R}_+, E)$ (except for the trivial $\mathcal{E} = \{\emptyset, E\}$) and so any other σ-algebra on $\mathbf{M}(\mathbb{R}_+, E)$ will not be suitable either. Note that it would be sufficient to treat the case where $E = \{0, 1\}$ endowed with its total σ-algebra. Nevertheless, a short appendix devoted to a similar question make us think that the situation is not really good in this direction.

In the same circle of ideas, we can weaken (H1), to get say (H′1), by replacing in the second point the σ-algebra $\mathcal{M}(\mathbb{R}_+, E)$ by the trace of $\mathcal{F}(\mathbb{R}_+, E)$ on $\mathbb{M}(\mathbb{R}_+, E)$, but this is done to the detriment of the admissible families of probabilities $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$, as it will then be more difficult to find such a Markovian family.

d) The typical occurence of the property (H1) is in the situation where E is a topological space, $\mathcal{E}$ is its Borelian σ-field and $\mathbb{M}(\mathbb{R}_+, E) = \mathbb{D}_+(\mathbb{R}_+, E)$, the set of all càd trajectories from $\mathbb{R}_+$ to E (eg in [6], E was a Polish space and $\mathbb{M}(\mathbb{R}_+, E)$ was $\mathbb{D}(\mathbb{R}_+, E)$ the Skorokhod space of càdlàg paths).

The singletons give trivial examples of sets satisfying (H1), but they are not compatible with the initial condition parametrization for Markovian families, except if $\mathcal{E} = \{\emptyset, E\}$!

Nevertheless even if there is no topology corresponding to $\mathcal{E}$, there always exists a set $\mathbb{M}(\mathbb{R}_+, E)$ satisfying (H1) and on which one can put a Markovian family: the set consisting of trajectories $\omega \in \mathbf{M}(\mathbb{R}_+, E)$ for which there exist an increasing sequence $(t_i)_{i\geq 0}$ of elements of $\bar{\mathbb{R}}_+$, satisfying $t_0 = 0$ and $\lim_{i\to\infty} t_i = +\infty$, and a sequence $(x_i)_{i\geq 0}$ of elements of E, such that

$$\forall\ i \geq 0,\ \forall\ t_i \leq s < t_{i+1}, \qquad X_s(\omega) \;=\; x_i$$

But let us give a more exotic example of path space satisfying (H′1) (and not (H1)): we take $E =]0, 1[$, $\mathcal{E} = \mathcal{R}_{]0,1[}$ and $\mathbb{M}(\mathbb{R}_+, E)$ is the set of trajectories $\omega \in \mathbf{M}(\mathbb{R}_+,]0, 1[)$ verifying

$$\forall\ t \geq 0, \qquad X_t(\omega) \;=\; \lim_{s\to 0_+} \frac{1}{s}\int_t^{t+s} X_u(\omega)\, du$$

The above examples also illustrate how the first requirement of (H1) can be seen as asking for the "regularity" of the paths to have a "local" feature.

e) To the Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$, we can associate the semigroup $(P_{s,t})_{0\leq s\leq t}$, whose elements act on $\mathcal{B}_{\mathrm{b}}(E)$ via the formulae:

$$\forall\ 0 \leq s \leq t,\ \forall\ \varphi \in \mathcal{B}_{\mathrm{b}}(E),\ \forall\ x \in E, \qquad P_{s,t}(\varphi)(x) \;=\; F_{t,0,\varphi}(s, x)$$

Due to our definition of the σ-algebras $\mathcal{M}([t, +\infty[, E)$, for $t \geq 0$, it is classical to see that the semigroup determines the Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$.

For all $T \geq 0$, we will denote the vector space

$$\mathcal{B}_T \stackrel{\text{def.}}{=} \{F_{T,0,\varphi}\ ;\ \varphi \in \mathcal{B}_{\mathrm{b}}(E)\} \quad \subset \quad \mathcal{A}_T$$

Now let $(\widetilde{\mathbb{P}}_{t,x})_{t\geq 0,\,x\in E}$ be another Markovian family whose time-space generators are the $(\widetilde{\mathcal{A}}_T, \widetilde{L}_T)$, for $T \geq 0$. As a consequence of the above discussion, if we assume that for all $T \geq 0$, $(\widetilde{\mathcal{A}}_T, \widetilde{L}_T)$ is an extension of $(\mathcal{B}_T, L_T)$, ie $\mathcal{B}_T \subset \widetilde{\mathcal{A}}_T$ and $\widetilde{L}_{T|\mathcal{B}_T} = L_{T|\mathcal{B}_T}$, then we have $(\widetilde{\mathbb{P}}_{t,x})_{t\geq 0,\,x\in E} = (\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$.

In particular, the family $(\mathcal{A}_T, L_T)_{T\geq 0}$ is caracteristic of $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$, ie the abstract martingale problems uniquely determine the Markovian family.

f) The definition of a Markovian family and the condition (H2) could also be considered with respect to other σ-fields verifying the second point of (H1) on our sets of paths (in which cases one has in addition to assume the measurability of the mapping appearing in the remark (a) above). But taking into account our particular choice of $\mathcal{M}(\mathbb{R}_+, E)$ and monotonous class arguments, they can be simplified and expressed through the associated semigroup: for instance the condition (H2) is equivalent to

(H′2) *For all $T \geq 0$ and all $A \in \mathcal{E}$, the mapping*

$$[0, T] \times E \ni (t, x) \mapsto P_{t,T}(\mathbf{1}_A)(x)$$

is $\mathcal{R}_{[0,T]} \otimes \mathcal{E}$-measurable.

Thus in this situation it appears that the role of the particular path $\diamond$ entering in the definition of the injections I_t, for $t \geq 0$, is not very important: for instance (H2) would not have been affected if we had chosen to let $\diamond$ depend in a measurable way on $X_t(\omega)$ (eg we could have rather considered for any $t \geq 0$ the injection defined by $X_s \circ I_t = X_{s\vee t}$ for every $s \geq 0$, except that it is not so natural to assume that $\mathbb{M}(\mathbb{R}_+, E)$ contains all constant paths, as we will see it in section 5).

g) The role of $T > 0$ in the definition of the generator $(\mathcal{A}_T, L_T)$ is not innocent: in the same way, we could have considered $(\mathcal{A}, L)$ the generator acting on measurable and locally bounded functions defined on $\mathbb{R}_+ \times E$ for which the martingale problems are satisfied, but it can be shown (for instance in the case of the real Brownian motion) that $(\mathcal{A}_T, L_T)$ can be a strict extension of the natural restriction of $(\mathcal{A}, L)$ on $\mathcal{B}_{\mathrm{b}}([0, T] \times E)$. Also note that there are some links between $(\mathcal{A}, L)$ and the full generators defined in [9], but they are not strictly the same, in particular due to the inhomogeneity in time.

h) A traditional object in related set-ups is the family $(\theta_t)_{t\geq 0}$ of the time shifts acting on $\mathbf{M}(\mathbb{R}_+, E)$, which are defined by

$$\forall\, t, s \geq 0,\ \forall\, \omega \in \mathbf{M}(\mathbb{R}_+, E), \qquad X_s(\theta_t(\omega)) \;=\; X_{t+s}(\omega)$$

and more precisely, for $t \geq 0$ given, θ_t is a measurable map from $(\mathbf{M}([t, +\infty[, E), \sigma(X_s\ ;\ s \geq t))$ to $(\mathbf{M}(\mathbb{R}_+, E), \sigma(X_s\ ;\ s \geq 0))$. But with our definition of the $\mathbb{M}([t, +\infty[, E)$, for $t \geq 0$, it is not clear that the image of $\mathbb{M}([t, +\infty[, E)$ under θ_t is included into $\mathbb{M}(\mathbb{R}_+, E)$ (eg if the random variables X_t naturally take values in different subsets of E as $t \geq 0$ varies, see for instance the end of this remark).

Nevertheless, if we assume in addition that for all $t \geq 0$, $\theta_t(\mathbb{M}([t, +\infty[, E)) \subset \mathbb{M}(\mathbb{R}_+, E)$, then (H2) is easily seen to be equivalent to

(H″2) *For all $A \in \mathcal{M}(\mathbb{R}_+, E)$ the mapping*

$$\mathbb{R}_+ \times E \ni (t, x) \mapsto \mathbb{P}_{t,x}[\theta_t^{-1}(A)]$$

is $\mathcal{R}_+ \otimes \mathcal{E}$-measurable.

Note that this hypothesis is just asking for the time-homogeneous Markov process $(t, X_t)_{t\geq 0}$ "with sufficiently regular trajectories" to admit a measurable kernel of transition probabilities from $\mathbb{R}_+ \times E$ to $\mathbb{M}(\mathbb{R}_+, \mathbb{R}_+ \times E) \stackrel{\text{def.}}{=} \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+) \times \mathbb{M}(\mathbb{R}_+, E)$. So under the condition (H$''$2), there is no lost of generality to restrict ourself to the time-homogeneous case, for which (H2) is automatically fulfilled. This may seem as a very mild assumption, but one has sometimes to be careful about conditioning in just measurable settings, because of the lack of "regular" version (in fact our hypothesis on the Markov family consists in assuming the existence of regular conditional expectations, as we cannot deduce it from properties of the state space, and we will be able to construct every other conditional distributions we will need in terms of these ones). In the same spirit, recall that every stochastic process can be seen as an homogeneous Markov process, if the state space is sufficiently enlarged, so one can extend our setting to more general situations if one is able to check the existence of a measurable version of conditional probabilities (but in general this regularity property requires more structure on the new state space which is now a set of paths), see the example of development presented in section 5. Nevertheless, we think that the point of view of inhomogeneous Markov processes is enough rewarding.

i) There is a more interesting generalization of our setting, which we will only mention: it corresponds to the cases where the differential ds in formula (1) is replaced by $da_s(\omega)$, for $\omega \in \mathbb{M}(\mathbb{R}_+, E)$, where $(a_s(\omega))_{s\geq 0}$ is a adapted continuous additive functional (with no martingale part). The martingale problems and consequently the definition of the $(\mathcal{A}_T, L_T)$, for $T \geq 0$, have to be changed accordingly.

One particular example would be to add to ds some local times, as those appearing when one is considering Euclidean diffusion processes (with regular coefficients) reflected on the boundary of a smooth domain. But as an intermediate step, one can replace ds by any (deterministic) nonnegative diffuse Radon measure on $\mathbb{R}_+$ (atomic cases would also be interesting, but certainly not so immediate).

j) Finally let us note that the corresponding discrete time problem can be imbedded in our setting: there everything starts with a time-inhomogeneous family of transition probabilities $(P_n)_{n\geq 0}$ on a mesurable space $(E, \mathcal{E})$ and a family $(g_n)_{n\geq 0}$ of functions belonging to $\mathcal{B}_{\mathrm{b}}(E)$ and satisfying $g_n \geq 1$ for all $n \geq 0$. Then one is interested in estimating the probability defined for any $n \geq 0$ by

$$\eta_n(\varphi) \quad \stackrel{\text{def.}}{=} \quad \frac{\mathbb{E}_{\eta_0}\left[\varphi(X_n)\prod_{0\leq m\leq n-1} g_m(X_m)\right]}{\mathbb{E}_{\eta_0}\left[\prod_{0\leq m\leq n-1} g_m(X_m)\right]}$$

where $(X_m)_{m\geq 0}$ is a Markov chain whose transition are given by the family $(P_m)_{m\geq 0}$ and whose initial law is a chosen probability η_0 on $(E, \mathcal{E})$, and where $\varphi \in \mathcal{B}_{\mathrm{b}}(E)$ is just a test function.

There is no problem in constructing an associated Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ on the third example of set $\mathbb{M}(\mathbb{R}_+, E)$ presented in above remark (d) which corresponds to the operation

$$(X_n)_{n\geq 0} \in E^{\mathbb{N}} \quad \mapsto \quad (X_{\lfloor t \rfloor})_{t\geq 0} \in \mathbb{M}(\mathbb{R}_+, E)$$

where $\lfloor \cdot \rfloor$ denote the integer part.

Let us also introduce the function

$$\forall\, t \geq 0,\, x \in E, \qquad U(t,x) \quad = \quad \ln(g_{\lfloor t \rfloor}(x))$$

then it appears that for $n \in \mathbb{N}$, the measure η_n is also given by (1), so we can just use the following considerations to device an efficient algorithm and to derive estimates on it. But in [4] we have presented a related direct discrete time approach.

2.2 Bounded perturbations of generators

In the last subsection, we have presented a way to associate to any Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$ an abstract generator $(\mathcal{A}_T, L_T)$, for all given $T\geq 0$. Here we will show how one can add some bounded operators to this generator, and we will study the perturbations induced by this kind of manipulations. If it was not for the generality of our setting, these would be standard results (cf for instance [1] or [9]), but in our situation we have to be a little more careful. There are two main motivations for these considerations:

- They give another family of simple and useful examples of functions belonging to $\mathcal{A}_T$.

- They will enable us to construct the approximating interacting particle systems in subsection 2.4 and to deduce some of their interesting features.

So again we consider a Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$ and for $T\geq 0$, let $(\mathcal{A}_T, L_T)$ be its associated generator on $[0,T]\times E$. Let $\widehat{R}$ be a locally bounded nonnegative kernel from $\mathbb{R}_+\times E$ to E, which is a mapping $(\mathbb{R}_+\times E)\times\mathcal{E}\to\mathbb{R}_+$ such that

- for all given $(t,x)\in\mathbb{R}_+\times E$, the application $\mathcal{E}\ni A\mapsto\widehat{R}((t,x),A)$ is a nonnegative measure
- for all fixed $A\in\mathcal{E}$, the function $\mathbb{R}_+\times E\ni(t,x)\mapsto\widehat{R}((t,x),A)$ is measurable
- for all $T\geq 0$, we have

$$\sup_{(t,x)\in[0,T]\times E} r(t,x) \quad < \quad +\infty$$

where for every $(t,x)\in[0,T]\times E$, we took $r(t,x)=\widehat{R}((t,x),E)=\max_{A\in\mathcal{E}}\widehat{R}((t,x),A)$.

Sometimes, we will write $\widehat{R}(t,x)$ for the measure $\mathcal{E}\ni A\mapsto\widehat{R}(t,x,A)\in\mathbb{R}_+$.

To such a kernel we can associate the operator R on $\mathcal{B}_{\mathrm{b}}(\mathbb{R}_+\times E)$ (which should be seen as a locally bounded time-inhomogeneous family of generators on $\mathcal{B}_{\mathrm{b}}(E)$, under the interpretation of $\widehat{R}(t,x,A)$ as the intensity of the occurence of a jump from $x\in E$ to $A\in\mathcal{E}$ at time $t\geq 0$, at least if $x\notin A$) defined by

$$\forall\ f\in\mathcal{B}_{\mathrm{b}}(\mathbb{R}_+\times E),\forall\ (t,x)\in\mathbb{R}_+\times E,\qquad R(f)(t,x)\quad=\quad\int f(t,y)\,\widehat{R}((t,x),dy)-r(t,x)f(t,x)$$

For $T\geq 0$, we will designate by R_T the natural restriction of R on $\mathcal{B}_{\mathrm{b}}([0,T]\times E)$.

Our first objective is to construct a Markovian family $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0,\,x\in E}$ such that for all $T\geq 0$, its generator $(\widehat{\mathcal{A}}_T,\widehat{L}_T)$ is an extension of $(\mathcal{A}_T, L_T+R_T)$.

To do so, we begin by considering homogeneous Markov chains on $\bar{\mathbb{R}}_+\times\mathbb{M}(\mathbb{R}_+,E)$, endowed with its natural σ-algebra, whose transition probability kernel $\check{P}$ is defined by

$$\forall\ (t,\omega)\in\mathbb{R}_+\times\mathbb{M}(\mathbb{R}_+,E),\ \forall\ I\in\mathcal{R}_+,\ \forall\ A\in\mathcal{M}(\mathbb{R}_+,E),$$

$$\check{P}((t,\omega),I\times A)\ =\ \int_{\mathbb{R}_+\times E}\mathbf{I}_I(t+s)\exp\left(-\int_0^s r(t+u,X_{t+u}(\omega))\,du\right)\widehat{R}((t+s,X_{t+s}(\omega)),dy)\mathbb{P}_{t+s,y}[A]\,ds$$

(the mass of $\check{P}((t,\omega),\mathbb{R}_+\times\mathbb{M}(\mathbb{R}_+,E))$ which maybe missing to 1 is reported to $(+\infty,\diamond)$), and by setting

$$\check{P}((+\infty,\diamond),\,\cdot\,)\quad=\quad\delta_{(+\infty,\diamond)}(\,\cdot\,)$$

Due to our hypotheses, especially the second point of (H1) and (H2), there is no real problem in verifying the measurability properties traditionally assumed for a probability kernel.

Then according to the theorem of Ionescu Tulcea (cf for instance [14]), for all $(t,\omega)\in\mathbb{R}_+\times\mathbb{M}(\mathbb{R}_+,E)$ there exists a unique probability $\check{\mathbb{P}}_{t,\omega}$ on $(\bar{\mathbb{R}}_+\times\mathbb{M}(\mathbb{R}_+,E))^{\mathbb{N}}$ (endowed with its natural product σ-field) under which the canonical coordinate chain is Markovian with $\check{P}$ as transition probability kernel and starts from the initial distribution $\delta_{(t,\omega)}$. Furthermore, for all measurable subset $A\subset(\bar{\mathbb{R}}_+\times\mathbb{M}(\mathbb{R}_+,E))^{\mathbb{N}}$, the mapping

$$\mathbb{R}_+\times\mathbb{M}(\mathbb{R}_+,E)\ni(t,\omega)\quad\mapsto\quad\check{\mathbb{P}}_{t,\omega}[A]$$

is also measurable.

Next we will transform this regular family $(\check{\mathbb{P}}_{t,\omega})_{t\geq 0,\,\omega\in\mathbb{M}(\mathbb{R}_+,E)}$ into the wanted Markov family $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0,\,x\in E}$ by transporting it through the following mapping. In order to be precise, we have to introduce some more notations:

Let $\check{E}$ be the set of elements $x=(t_i,y_i)_{i\geq 0}\in(\bar{\mathbb{R}}_+\times\mathbb{M}(\mathbb{R}_+,E))^{\mathbb{N}}$ such that there exists an index $0<i_\infty\leq\infty$ satisfying

$$\forall\ i\geq 0,\qquad i<i_\infty\ \Rightarrow\ t_i\ <\ t_{i+1}\quad\text{ and }\quad i\geq i_\infty\ \Rightarrow\ (t_i,y_i)=(+\infty,\diamond)$$

We endow this set with the σ-algebra $\check{\mathcal{E}}$ inherited from $(\mathcal{R}_{[0,+\infty]}\otimes\mathcal{M}(\mathbb{R}_+,E))^{\otimes\mathbb{N}}$.

Note that each $\check{\mathbb{P}}_{t,\omega}$, for $t\in\mathbb{R}_+$ and $\omega\in\mathbb{M}(\mathbb{R}_+,E)$, is in fact a probability on $(\check{E},\check{\mathcal{E}})$, and with this point of view, the family $(\check{\mathbb{P}}_{t,\omega})_{t\geq 0,\,\omega\in\mathbb{M}(\mathbb{R}_+,E)}$ obviously retains the same measurability regularity.

On the other hand we can define on this domain the mapping $\Phi\,:\check{E}\to\mathbb{M}(\mathbb{R}_+,E)$ given by

$$\forall\ x\in\check{E},\,\forall\ i\geq -1,\,\forall\ t_i\leq s<t_{i+1},\qquad X_s(\Phi(x))\ =\ X_s(y_i)$$

where we have used the same notations as before for elements of $\check{E}$, and with the convention that $t_{-1}=0$ and x_{-1} is the a priori fixed path $\diamond$.

Due to the first point of hypothesis (H1), $\Phi(x)$ really lives in $\mathbb{M}(\mathbb{R}_+,E)$. This mapping Φ is also clearly seen to be measurable.

Now let us define for all $(t,x)\in\mathbb{R}_+\times E$,

$$\widehat{\mathbb{P}}_{t,x}\ =\ \Phi(\check{\mathbb{P}}_{t,x})$$

where the probability $\check{\mathbb{P}}_{t,x}$ on $\check{E}$ is given by

$$\forall\ A\in\check{\mathcal{E}},\qquad\check{\mathbb{P}}_{t,x}(A)\ =\ \int\check{\mathbb{P}}_{t,\omega}(A)\,\mathbb{P}_{t,x}(d\omega)$$

Remark that for $t\geq 0$ and in the sense of the injection I_t, $\widehat{\mathbb{P}}_{t,x}$ can be seen as a probability on $\mathbb{M}([t,+\infty[,E)$.

It is time to verify that the family obtained by putting together these probabilities will do the job for which it was designed:

Proposition 2.3 *The family $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0,\,x\in E}$ is Markovian and satisfies (H2). Furthermore for any fixed $T\geq 0$, we have $\mathcal{A}_T\subset\widehat{\mathcal{A}}_T$ and for all $f\in\mathcal{A}_T$, $\widehat{L}_T(f)=L_T(f)+R_T(f)$.*

Proof:

The measurability requirements (in particular (H2)) follow from the above considerations.

But unlike the approach followed by Ethier and Kurtz in [9], here it is not sufficient to consider the underlying martingale problems (ie to merely prove the second part of the proposition) to insure the validity of the Markov property in the general way we have defined it; we are only allowed to play with the very basic objects we have just introduced. That is why the subsequent proof is quite too long and should be admitted at a first reading !

So we are wondering if for all $0 \leq t < s$, all $x \in E$, all $A \in \mathcal{M}([t,s],E)$ and all $B \in \mathcal{M}(]s,+\infty[,E)$, we have

$$\widehat{\mathbb{P}}_{t,x}[A \cap B] \quad = \quad \widehat{\mathbb{E}}_{t,x}[\mathbf{1}_A \widehat{\mathbb{P}}_{s,X_s}[B]]$$

Let us denote $\check{A} = \Phi^{-1}(A)$ and $\check{B} = \Phi^{-1}(B)$. Clearly it is equivalent to show that

$$\check{\mathbb{P}}_{t,x}[\check{A} \cap \check{B}] \quad = \quad \check{\mathbb{E}}_{t,x}[\mathbf{1}_{\check{A}} \widehat{\mathbb{P}}_{s,X_s \circ \Phi}[B]] \tag{7}$$

To show these equalities are true, we consider for fixed $s \geq 0$, the function

$$\begin{aligned} H_s \,:\, \bar{\mathbb{R}}_+ \times \mathbb{M}(\mathbb{R}_+, E) \quad &\to \quad [0,s] \times \mathbb{M}([0,s],E) \times [s,+\infty] \times \mathbb{M}([s,+\infty[,E) \\ (u,\omega) \quad &\mapsto \quad (s \wedge u, (X_t(\omega))_{0 \leq t \leq s}, s \vee u, (X_t(\omega))_{s \leq t}) \end{aligned}$$

We denote by $(\check{Z}_n)_{n\geq 0}$ the canonical coordinates on $\check{E}$ and next we naturally write for $n \geq 0$,

$$(T_n, Z_n, T'_n, Z'_n) \quad \stackrel{\text{def.}}{=} \quad H_s(\check{Z}_n)$$

Let us also define the integer variable

$$N \quad = \quad \inf\{n \geq 0 \,:\, T_n = s\}$$

which is $\check{\mathbb{P}}_{t,x}$-a.s. finite under our local boundedness condition on U.

Then we remark (eg by applying the monotonous class theorem) that at least on $\{N \geq 1\}$,

$$\begin{aligned} \check{A} \quad &\in \quad \sigma(\check{Z}_{n\wedge(N-1)}\,;\, n \geq 0) \\ \check{B} \quad &\in \quad \sigma(\check{Z}_{n+N-1}\,;\, n \geq 0) \end{aligned}$$

and more precisely that there exist $\breve{A} \in (\mathcal{R}_{[0,s]} \otimes \mathcal{M}([0,s],E))^{\otimes\mathbb{N}}$ and $\breve{B} \in (\mathcal{R}_{[s,+\infty[}\otimes \mathcal{M}(]s,+\infty[,E))^{\otimes\mathbb{N}}$ such that

$$\begin{aligned} \check{A} \quad &= \quad \{(T_{n\wedge(N-1)}, Z_{n\wedge(N-1)})_{n\geq 0} \in \breve{A}\} \\ \check{B} \quad &= \quad \{(T'_{n+N-1}, Z'_{n+N-1})_{n\geq 0} \in \breve{B}\} \end{aligned}$$

Unfortunately the fact that $N-1$ is not a stopping time prevents us from applying directly the strong Markov property for $(\check{Z}_n)_{n\geq 0}$ under $\check{\mathbb{P}}_{t,x}$.

But the interest of the previous objects is that the chain $(Z'_{n-1}, T'_n, Z_n, T_{n+1})_{n\geq 0}$ is also Markovian under $\check{\mathbb{P}}_{t,x}$, with the convention that $Z'_{-1} = \diamond$ (also identified with its restriction to the interval $[s,+\infty[$). More precisely, for fixed $0 \leq t < s$, its initial distribution is $\delta_\diamond \otimes \delta_s \otimes m_{t,x}$, where $m_{t,x}$ is the probability defined on $\mathbb{M}([0,s],E) \times [0,s]$ by

$\forall\, C \in \mathcal{M}([0,s],E)$, $\forall\, I \in \mathcal{R}_{[0,s]}$,

$$\begin{aligned} m_{t,x}(C \times I) \quad = \quad & \mathbb{P}_{t,x}\Big[\mathbf{1}_C((X_v)_{0\leq v\leq s})\Big\{\int_t^s \mathbf{1}_I(w) r(w,X_w) \exp\Big(-\int_t^w r(w',X_{w'})\,dw'\Big)\,dw \\ & +\mathbf{1}_I(s) \exp\Big(-\int_t^s r(w,X_w)\,dw\Big)\Big\}\Big] \end{aligned}$$

and we calculate down that its probability transition kernel $\check{P}$ satisfies

$\forall\, (z',u',z,u)\in\mathbb{M}([s,+\infty[,E)\times[s,+\infty[\times\mathbb{M}([0,s],E)\times[0,s]$, $\forall\,(C',I',C,I)\in\mathcal{M}([s,+\infty[,E)\times\mathcal{R}_{[s,+\infty[}\times\mathcal{M}([0,s],E)\times\mathcal{R}_{[0,s]}$,

$$\begin{aligned}
\breve{P}((z',u',z,u),C'\times I'\times C\times I) \quad &= \\
&\mathbf{I}_{\{u<s\}}\Big[\mathbb{P}_{s,X_s(z)}[C']\mathbf{I}_{I'}(s)\int_E \widetilde{R}(u,X_u(z),dy)\mathbb{E}_{u,y}\Big[\mathbf{I}_C((X_v)_{0\leq v\leq s})\Big\{\mathbf{I}_I(s)\exp\Big(-\int_u^s r(w,X_w)\,dw\Big) \\
&+\int_{[u,s[}\mathbf{I}_I(w)r(w,X_w)\exp\Big(-\int_u^w r(w',X_{w'})\,dw'\Big)\,dw\Big\}\Big]\Big] \\
&+\mathbf{I}_{\{u=s=u'\}}\Big[\mathbb{E}_{s,X_s(z)}\Big[\mathbf{I}_{C'}((X_{v'})_{v'\geq s})\int_{]s,+\infty[\cap I'} r(v,X_v)\exp\Big(-\int_s^v r(w,X_w)\,dw\Big)\,dv\Big]\mathbf{I}_C(\diamond)\mathbf{I}_I(s)\Big] \\
&+\mathbf{I}_{\{u=s,u'>s\}}\Big[\int_E \widetilde{R}(u',X_{u'}(z'),dy)\mathbb{E}_{u',y}\Big[\mathbf{I}_{C'}((X_{v'})_{v'\geq s})\int_{]u',+\infty[\cap I'} r(v,X_v) \\
&\exp\Big(-\int_{u'}^v r(w,X_w)\,dw\Big)\,dv\Big]\mathbf{I}_C(\diamond)\mathbf{I}_I(s)\Big]
\end{aligned}$$

where the new probability kernel $\widetilde{R}$ from $\mathbb{R}_+\times E$ to E is given by the renormalization

$$\forall\, u\geq 0,\,\forall\, x\in E,\qquad \widetilde{R}(u,x) \;=\; \begin{cases}\widehat{R}(u,x)/r(u,x) & ,\ \text{if } r(u,x)>0\\ \delta_\diamond & ,\ \text{otherwise}\end{cases}$$

(note that $\breve{\mathbb{P}}_{t,x}$-a.s. and for any $n\geq 0$, either $T_{n+1}=+\infty$ or $r(Z_n,T_{n+1})>0$), and where as usual the possible missing mass is put on $(\diamond,+\infty,\diamond,s)$, which is also assumed to be a cemetery point.

Let us denote for $(z',u',z,u)\in\mathbb{M}([s,+\infty[,E)\times[s,+\infty[\times\mathbb{M}([0,s],E)\times[0,s]$, $\breve{\mathbb{P}}_{z',u',z,u}$ the law of a Markov chain $(\breve{Z}'_n,\breve{T}'_n,\breve{Z}_n,\breve{T}_n)_{n\geq 0}$ starting from (z',u',z,u) and whose kernel is $\breve{P}$. Then we are in position to apply the strong Markov property to the stopping time $N-1$ with respect to the chain $(Z'_{n-1},T'_n,Z_n,T_{n+1})_{n\geq 0}$, and we get for $x\in E$ and $0\leq t<s$ (which also insures that $\breve{\mathbb{P}}_{t,x}$-a.s. $N\geq 1$),

$$\begin{aligned}
\breve{\mathbb{P}}_{t,x}[\breve{A}\cap\breve{B}] &= \breve{\mathbb{E}}_{t,x}[\mathbf{I}_{\breve{A}}((T_{n\wedge(N-1)},Z_{n\wedge(N-1)})_{n\geq 0})\mathbf{I}_{\breve{B}}((T'_{n+N-1},Z'_{n+N-1})_{n\geq 0})] \\
&= \breve{\mathbb{E}}_{t,x}[\mathbf{I}_{\breve{A}}((T_{n\wedge(N-1)},Z_{n\wedge(N-1)})_{n\geq 0})\breve{\mathbb{P}}_{Z'_{N-2},s,Z_{N-1},s}[\mathbf{I}_{\breve{B}}((\breve{T}'_n,\breve{Z}'_{n+1})_{n\geq 0})]]
\end{aligned}$$

But we notice that for all $(z,z')\in\mathbb{M}([0,s],E)\times\mathbb{M}([s,+\infty[,E)$, the law of $(\breve{T}'_n,\breve{Z}'_{n+1})_{n\geq 0}$ under $\breve{\mathbb{P}}_{z',s,z,s}$ is $\breve{\mathbb{P}}_{s,X_s(z)}$, so (7) follows from the fact that $\breve{\mathbb{P}}_{s,X_s(z)}[\breve{B}]=\widehat{\mathbb{P}}_{s,X_s(z)}[B]$ and from the $\breve{\mathbb{P}}_{t,x}$-a.s. equality $X_s\circ\Phi=X_s(Z_{N-1})$.

Now that we have shown that $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0,x\in E}$ is a Markovian family, it remains to verify the affirmation about the generators.

We first remark that due to the above result, it is sufficient to show that for all fixed $0\leq t\leq s\leq T$, all fixed $x\in E$ and all $f\in\mathcal{A}_T$,

$$\widehat{\mathbb{E}}_{t,x}\left[f(s,X_s)-f(t,X_t)-\int_t^s \widehat{L}_T(f)(u,X_u)\,du\right] \;=\; 0 \tag{8}$$

If in addition we knew that the mapping $r:\mathbb{R}_+\times E\to\mathbb{R}_+$ is constant, we could transpose the usual arguments given by Ethier and Kurtz in the proof of the proposition 10.2 p. 256 of [9] to verify this equality (their processes are assumed to be càdlàg so they are allowed to use the notation X_{u-}, and in our setting this has to be interpreted in the following sense: let us come back to the notations introduced before proposition 2.3 and consider $x\in\breve{E}$, if there exists $1\leq i<i_\infty$ such that $u=t_i$, then we denote $X_{u-}(\Phi(x))=X_u(y_{i-1})$, otherwise we take $X_{u-}(\Phi(x))=X_u(\Phi(x))=X_u(y_i)\,\ldots$).

But it is well-known that the general situation can be reduced to the previous case via an acceptation/rejection procedure: at each of more frequently selected times, there is more probability that the process stay at the present position, so these instants are only proposed jump times. We begin by noting that the law of $(X_u)_{t\leq u\leq s}$ under $\widehat{\mathbb{P}}_{t,x}$ and the values of $\widehat{L}_T(f)(u,y)$, for $t\leq u\leq s$ and $y\in E$, only depend on the restriction of $\widehat{R}$ on $[t,s]\times E\times\mathcal{E}$, so to prove (8), we can assume that

$$r \stackrel{\text{def.}}{=} \sup_{(u,y)\in\mathbb{R}_+\times E} r(u,y) \;<\; +\infty$$

Under this extra assumption, we construct a new bounded kernel $\widehat{R}'$ from $\mathbb{R}_+\times E$ to E via the formula

$$\forall\,(u,y)\in\mathbb{R}_+\times E,\qquad \widehat{R}'(u,y) \;=\; \widehat{R}(u,y)+(r-r(u,y))\delta_y$$

This kernel admits the required regularity conditions and the convenient property that

$$\forall\,u\geq 0,\,\forall\,y\in E,\qquad r'(u,y) \;=\; r$$

meanwhile its associated operators are the same as the previous ones:

$$\forall\,T\geq 0,\qquad R'_T \;=\; R_T$$

Now as before, we can construct from $\widehat{R}'$ and $(\mathbb{P}_{u,y})_{(u,y)\in\mathbb{R}_+\times E}$ the Markovian family $(\widehat{\mathbb{P}}'_{u,y})_{(u,y)\in\mathbb{R}_+\times E}$ and according to the previous case, we have

$$\begin{aligned} \mathcal{A}_T &\subset \widehat{\mathcal{A}}'_T \\ \forall\,f\in\mathcal{A}'_T,\qquad \widehat{L}'_T(f) &= L_T(f)+R_T(f) \end{aligned}$$

so (8) will be proved if we can show that $\widehat{\mathbb{P}}'_{t,x}=\widehat{\mathbb{P}}_{t,x}$.

But this is a classical computation based on one hand on the fact that for the construction of the $\widehat{\mathbb{P}}_{u,y}$, for $t\leq u\leq s$ and $y\in E$, the difference of the proposed jump times are mutually independent, independent of the trajectories between these proposed times and distributed as exponential variables of parameter r, and on the other hand on the following elementary observation:

Lemma 2.4 *Let $(\tau_n)_{n\geq 1}$ be a sequence of independent exponential random variables of parameter r and let $(V_n)_{n\geq 1}$ be a sequence of independent uniform random variables on [0,1], both families are furthermore assumed to be independent of each other. Let $g:\mathbb{R}_+\to[0,1]$ be a given measurable mapping.*

We note

$$\begin{aligned} N &= \inf\{n\geq 1\,:\,V_n\leq g(\tau_1+\cdots+\tau_n)\} \;\leq\; +\infty \\ T &= \sum_{1\leq n<N+1} T_n \;\leq\; +\infty \end{aligned}$$

then the distribution of T is defined by

$$\forall\,u\geq 0,\qquad \mathbb{P}[T>u] \;=\; \exp\left(-\int_0^u g(v)\,dv\right)$$

This result is applied, for any fixed $t \leq u \leq s$ and any trajectory $\omega \in \mathbb{M}([u, +\infty[, E)$, with $g : \mathbb{R}_+ \ni v \mapsto r(u+v, X_{u+v}(\omega))/r$, but the easy proofs are left to the reader (one has just to take into account the fact that at the proposed jump times "corresponding" to the mass $r - r(u, y)$, the process remains at the same place, so one is able to use the Markov property at these times for the familly $(\mathbb{P}_{u,y})_{u\geq 0,\, y\in E})$.

■

Remark 2.5: The above reduction to the case where $r(\,\cdot\,,\,\cdot\,)$ is a constant function is often useful in practice, because independent exponential waiting times are easy to simulate. This approach is also more convenient when we will have to resort to coupling arguments.

If the mapping $r(\,\cdot\,,\,\cdot\,)$ was a priori assumed to be bounded, this procedure could have been used for the whole time interval $\mathbb{R}_+$ (instead of locally) for the construction of $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0, x\in E}$. In fact, as we will essentially work with a finite horizon $T > 0$ in what follows, we could have merely taken into account this description (and then the theorem of Ionescu Tulcea is not really necessary, since only random but finite numbers of selections are needed, we can just use the theorem of integration of kernels, see for instance [7]). Nevertheless, for completeness we have prefered to present a construction valid for all times (note that we are not under the hypotheses of Kolmogorov's theorem, so it is not sufficient to construct compatible processes on finite time intervals to get one on $\mathbb{R}_+$), then, except in some very particular situations (eg if U is constant), the time interval between two consecutive proposed jump instants will not be an exponential random variable.

Note that we could also haven take advantage of this reduction in the first step of the proof for the Markov property, to reduce some formulae, but the argument would have remain unchanged, as opposed to considerations of martingale problems (the last step in the above proof, or next proposition), where to consider a constant r really simplify the analysis.

Now that we have seen that $(\widehat{\mathcal{A}}_T, \widehat{L}_T)$ is an extension of $(\mathcal{A}_T, L_T + R_T)$, for $T \geq 0$, we can go even further, because the reciproque is also true:

Proposition 2.6 *For all $T \geq 0$, we have*

$$(\widehat{\mathcal{A}}_T, \widehat{L}_T) \quad = \quad (\mathcal{A}_T, L_T + R_T)$$

Proof:

So let $T > 0$ be fixed and $f \in \widehat{\mathcal{A}}_T$ be given, we have to show that for all fixed $0 \leq t \leq s \leq T$ and all fixed $x \in E$,

$$\mathbb{E}_{t,x}\left[f(s, X_s) - f(t, X_t) - \int_t^s (\widehat{L}_T - R_T)(f)(u, X_u)\,du\right] \quad = \quad 0 \tag{9}$$

since it will then follow that $f \in \mathcal{A}_T$ and that $L_T(f) = \widehat{L}_T(f) - R_T(f)$.

This is a "local" result, so once again, we can assume that the mapping $r(\cdot, \cdot)$ is constant.

We will work under the probability $\check{\mathbb{P}}_{t,x}$, the random variable T_1 stands for the first proposed jump time appearing ($T_1 - t$ follows an exponential law of parameter r), X_{T_1-} is defined as it was at the end of the proof of proposition 2.3, and $\mathbb{P}_{t,x}$ is seen as the law of $X \stackrel{\text{def.}}{=} (X_u)_{u\geq t}$.

With these notations, we can write

$$\begin{aligned}(f(s, X_s) - f(t, X_t))\mathbb{1}_{T_1\leq s} \quad &= \quad (f(s, X_s) - f(T_1, X_{T_1}))\mathbb{1}_{T_1\leq s} + (f(T_1, X_{T_1}) - f(T_1, X_{T_1-}))\mathbb{1}_{T_1\leq s} \\ &\quad +(f(T_1, X_{T_1-}) - f(t, X_t))\mathbb{1}_{T_1\leq s}\end{aligned}$$

But we notice that by construction, X admits a strong Markov property with respect to the time T_1, so using the fact that $f \in \mathcal{A}_T$, we get

$$\begin{aligned}
\check{\mathbb{E}}_{t,x}[(f(s,X_s)-f(T_1,X_{T_1}))\mathbf{1}_{T_1\leq s}] &= \check{\mathbb{E}}_{t,x}[\mathbf{1}_{T_1\leq s}\check{\mathbb{E}}_{T_1,X_1}[f(s,X_s)-f(T_1,X_{T_1})]] \\
&= \check{\mathbb{E}}_{t,x}\left[\mathbf{1}_{T_1\leq s}\check{\mathbb{E}}_{T_1,X_1}\left[\int_{T_1}^{s}\widehat{L}_T(f)(u,X_u)\,du\right]\right] \\
&= \check{\mathbb{E}}_{t,x}\left[\int_{s\wedge T_1}^{s}\widehat{L}_T(f)(u,X_u)\,du\right]
\end{aligned}$$

On the other hand, it is quite clear that by the properties of T_1,

$$\begin{aligned}
\check{\mathbb{E}}_{t,x}[(f(T_1,X_{T_1})-f(T_1,X_{T_1-}))\mathbf{1}_{T_1\leq s}] &= \check{\mathbb{E}}_{t,x}[R_T(f)(T_1,X_{T_1-})\mathbf{1}_{T_1\leq s}]/r \\
&= \int_t^s \mathbb{E}_{t,x}[R_T(f)(u,X_u)]\exp(-r(u-t))\,du
\end{aligned}$$

and

$$\check{\mathbb{E}}_{t,x}[(f(T_1,X_{T_1-})-f(t,X_t))\mathbf{1}_{T_1\leq s}] = r\int_t^s \mathbb{E}_{t,x}[f(u,X_u)-f(t,X_t)]\exp(-r(u-t))\,du$$

Now let us denote for $t\leq s\leq T$, $g(s)\stackrel{\text{def.}}{=}\mathbb{E}_{t,x}[f(s,X_s)-f(t,X_t)]$, we have

$$\begin{aligned}
g(s) &= \frac{\check{\mathbb{E}}_{t,x}[(f(s,X_s)-f(t,X_t))\mathbf{1}_{T_1>s}]}{\check{\mathbb{P}}_{t,x}[T_1>s]} \\
&= \exp(r(s-t))(\check{\mathbb{E}}_{t,x}[f(s,X_s)-f(t,X_t)]-\check{\mathbb{E}}_{t,x}[(f(s,X_s)-f(t,X_t))\mathbf{1}_{T_1\leq s}]) \\
&= \exp(r(s-t))\Big(\check{\mathbb{E}}_{t,x}\left[\int_t^s\widehat{L}_T(f)(u,X_u)\,du\right]-\check{\mathbb{E}}_{t,x}\left[\int_{s\wedge T_1}^s\widehat{L}_T(f)(u,X_u)\,du\right] \\
&\quad -\int_t^s\mathbb{E}_{t,x}[R_T(f)(u,X_u)]\exp(-r(u-t))\,du-r\int_t^s g(u)\exp(-r(u-t))\,du\Big) \\
&= \exp(r(s-t))\Big(\check{\mathbb{E}}_{t,x}\left[\int_t^s\widehat{L}_T(f)(u,X_u)\mathbf{1}_{u\leq T_1}\,du\right] \\
&\quad -\int_t^s\mathbb{E}_{t,x}[R_T(f)(u,X_u)]\exp(-r(u-t))\,du-r\int_t^s g(u)\exp(-r(u-t))\,du\Big) \\
&= \exp(r(s-t))\Big(\int_t^s\mathbb{E}_{t,x}\left[\widehat{L}_T(f)(u,X_u)\right]\exp(-r(u-t))\,du \\
&\quad -\int_t^s\mathbb{E}_{t,x}[R_T(f)(u,X_u)]\exp(-r(u-t))\,du-r\int_t^s g(u)\exp(-r(u-t))\,du\Big) \\
&= \exp(r(s-t))\Big(\int_t^s\mathbb{E}_{t,x}\left[(\widehat{L}_T-R_T)(f)(u,X_u)\right]\exp(-r(u-t))\,du \\
&\quad -r\int_t^s g(u)\exp(-r(u-t))\,du\Big)
\end{aligned}$$

This differential equation satisfied by $\int_t^s g(u)\exp(-r(u-t))\,du$, for $t\leq s\leq T$, has a unique continuous solution, which is

$$\begin{aligned}
&\int_t^s g(u)\exp(-r(u-t))\,du \\
&\quad = \int_t^s\exp(r(u-s))\int_t^u\mathbb{E}_{t,x}\left[(\widehat{L}_T-R_T)(f)(v,X_v)\right]\exp(-r(v-t))\,dv\,du
\end{aligned}$$

and so using again the above equation, that can be easily rewritten in the form (9).

■

Let us give a first consequence of this identity, mentioned at the beginning of this subsection and which will be a powerful tool in the subsequent development (because it is the one which will enable us to remove all regularity assumptions). A little more precisely, as we will mainly work with martingales (and not directly with their increasing processes), we need to know a lot of them, and the following result is a good way to construct some interesting ones, via the description of new elements of $\mathcal{A}_T$.

Corollary 2.7 *Let $T > 0$ be fixed, and let $V \in \mathcal{B}_{\mathrm{b}}([0,T] \times E)$ and $\varphi \in \mathcal{B}_{\mathrm{b}}(E)$ be given. We consider the function defined by*

$$\forall\, 0 \leq t \leq T, \forall\, x \in E, \qquad G_{T,V,\varphi,\widehat{R}}(t,x) \;=\; \widehat{\mathbb{E}}_{t,x}\left[\exp\left(\int_t^T V_s(X_s)\,ds\right)\varphi(X_T)\right]$$

Then the mapping $G_{T,V,\varphi,\widehat{R}}$ belongs to $\mathcal{A}_T$, and we have

$$\forall\, 0 \leq t \leq T, \forall\, x \in E, \qquad L_T(G_{T,V,\varphi,\widehat{R}})(t,x) \;=\; -V_t(x)G_{T,V,\varphi,\widehat{R}}(t,x) - R(G_{T,V,\varphi,\widehat{R}})(t,x)$$

Proof:

This follows immediatly from the combination of lemma 2.1 and proposition 2.6.

■

Remark 2.8: As our primary object of interest, the flow $(\eta_t)_{t\geq 0}$ defined in (1), is only using the probability $\mathbb{P}_{\eta_0}$, we could have associated to any time $T \geq 0$ and any given initial probability η_0, another generator by only requiring that the process $(M_{0,t}(f))_{0\leq s\leq T}$ should be a martingale under $\mathbb{P}_{\eta_0}$ (see (4) for the notation).

But via the use of traditionally coupled processes and perturbation kernels making them jump on the diagonal, it is easy to find an example for which the proposition 2.6 relative to this slightly different notion is not true.

This comes from the cut and paste trajectories constructions (indeed the main reason for the first point of condition (H1)), which can lead the new process to explore more points of the state space than those which can be attainable under $\mathbb{P}_{\eta_0}$.

2.3 "Coupling" arguments

As we shall see in the subsequent development, it is sometimes useful to compare the initial Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ with its just constructed modification $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0,\, x\in E}$, at least in the cases where the perturbation $\widehat{R}$ is small, and one seemingly nice way to do it would be to couple them.

But once again our setting does not enable us to work it out in the traditional manner: for instance even if $\widehat{R} \equiv 0$, there may not exist the usual Markovian coupling of $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ with himself, which would satisfy that when the two coordinates coincide they stay together and so evolve identically, and when they are different they evolve independently, up to the (contingently) time when they would be equal. The basic obstruction is that in general it is not clear that the diagonal $\triangle(E) = \{(x,x) \in E^2 \,:\, x \in E\}$ belongs to $\mathcal{E} \otimes \mathcal{E}$. Recall that we have even not assumed that $(E, \mathcal{E})$ is separated, but to consider its natural diagonal

$$\bar{\triangle}(E) \;=\; \{(x,y) \in E^2 \,:\, \delta_x = \delta_y\}$$

would not have improved the situation.

In order to overcome this difficulty, we will only couple probabilities (ie we look for probabilities on a product space with specified marginals, eg $\mathbb{P}_{0,x}$ and $\widehat{\mathbb{P}}_{0,x}$, for a fixed $x \in E$) and not Markovian families; in fact our couplings will not be Markovian processes. This will not be important, because our purpose is to try to make processes issued from a same position to stay together the longest possible time, and not (as it is more customary) to attempt to make them come back together if they are separated.

We will also have resort to another trick, as we will really be interested in the diagonal of $E \times E$ for general measurable space $(E, \mathcal{E})$. If m is a nonnegative finite measure on $(E^2, \mathcal{E}^{\otimes 2})$, we will make the convention that

$$m(\triangle(E)) \stackrel{\text{def.}}{=} \sup m_1(E)$$

where the supremum (which is in fact a maximum) is taken over all nonnegative measures m_1 defined on $(E, \mathcal{E})$ such that $m \geq m_2$, where m_2 is the image of m_1 under the mapping $E \ni x \mapsto (x, x) \in E^2$.

This notion is quite natural, because the classical proof shows that if μ_1 and μ_2 are probabilities on $(E, \mathcal{E})$, then there exists a coupling m of them on $(E^2, \mathcal{E}^{\otimes 2})$ verifying

$$m(E^2) - m(\triangle(E)) = \frac{1}{2} \|\mu_1 - \mu_2\|_{\mathrm{tv}} \tag{10}$$

As we already mentioned it, we will consider here couplings of probabilities on path spaces, so we have to be a little more precise about path spaces associated to product state space: for $N \geq 2$, we will always take $\mathbb{M}(\mathbb{R}_+, E^N) = \mathbb{M}(\mathbb{R}_+, E)^N$, definition which implies that $\mathcal{M}(\mathbb{R}_+, E^N) = \mathcal{M}(\mathbb{R}_+, E)^{\otimes N}$, and so (H1) is clearly verified.

A typical example of the kind of results we are looking for is the following one, where the Markovian family $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0, x\in E}$ is constructed as in the previous section, starting from $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ and $\widehat{R}$:

Proposition 2.9 *Let $T \geq 0$ and $x \in E$ be given. Then there exists a coupling $\mathbb{P}^{\dagger}_{0,(x,x)}$ of $\mathbb{P}_{0,x}$ and $\widehat{\mathbb{P}}_{0,x}$ such that*

$$\mathbb{P}^{\dagger}_{0,(x,x)}[(X_t)_{0\leq t\leq T} \neq (\widehat{X}_t)_{0\leq t\leq T}] \leq 1 - \exp\left(-T \sup_{0\leq t\leq T,\, x\in E} \widehat{R}(t, x, E)\right)$$

(where $(X_t, \widehat{X}_t)_{t\geq 0}$ stands for the canonical coordinate process on $\mathbb{M}(\mathbb{R}_+, E^2)$, and by convention, we have taken $\mathbb{P}^{\dagger}_{0,(x,x)}[(X_t)_{0\leq t\leq T} \neq (\widehat{X}_t)_{0\leq t\leq T}] = 1 - \mathbb{P}_{0,(x,x)}[\triangle(\mathbb{M}([0,T], E))]$).

Proof:

The horizon $T \geq 0$ being fixed, it is sufficient to construct a coupling $\mathbb{P}^{\dagger}_{0,(x,x),[0,T]}$ on $\mathbb{M}([0,T], E^2)$ of the restrictions to $\mathbb{M}([0,T], E)$ of $\mathbb{P}_{0,x}$ and $\widehat{\mathbb{P}}_{0,x}$, satisfying the required condition, because it is then immediate to extend it to a coupling over the whole $\mathbb{M}(\mathbb{R}_+, E^2)$, by letting, after time T, the coordinates evolve independently and respectively according to $(\mathbb{P}_{t,x})_{t\geq T, x\in E}$ and $(\widehat{\mathbb{P}}_{t,x})_{t\geq T, x\in E}$.

This remark make it clear that there is no lost of generality to come down to the situation where the quantity $R(t, x, E)$ does not depend on $t \geq 0$ and $x \in E$, and where its common value is $r = \sup_{0\leq t\leq T,\, x\in E} R(t, x, E)$, in terms of the initial kernel.

So under this hypothesis, let us consider a “generalized” Markov family on $E^2 \times \{0, 1\}$, $(\mathbb{P}^{\ddagger}_{t,x})_{t\geq 0,\, x\in E^2\times\{0,1\}}$, in the sense that it will not verify the first assumption of initial parametrization property:

• For $(x,y) \in E^2$, the probability $\mathbb{P}^{\ddagger}_{t,(x,y,0)}$ is $\mathbb{P}^{\ddagger}_{t,(x,x)} \otimes \delta_{0_{[t,+\infty[}}$, where the first factor is the image of the probability $\mathbb{P}_{t,x}$ by the mapping

$$\mathbb{M}([t,+\infty[,E) \ni \omega \quad \mapsto \quad (\omega,\omega) \in \mathbb{M}([t,+\infty[,E^2)$$

and where for any $t \geq 0$ and any element a, $a_{[t,+\infty[}$ stands for the constant path defined over the time interval $[t,+\infty[$ and always taking the value a.

• For $(x,y) \in E^2$, the probability $\mathbb{P}^{\ddagger}_{t,(x,y,1)}$ is just the product

$$\mathbb{P}_{t,x} \otimes \widehat{\mathbb{P}}_{t,y} \otimes \delta_{1_{[t,+\infty[}}$$

We now perturbe this family by the nonnegative kernel $\widehat{R}^{\ddagger}$ from $\mathbb{R}_+ \times E^2 \times \{0,1\}$ to $E^2 \times \{0,1\}$, defined by

$$\forall\, t \geq 0, \forall\, (x,y,z) \in E^2 \times \{0,1\}, \qquad \widehat{R}^{\ddagger}(t,(x,y,z)) \;=\; \delta_x \otimes \widehat{R}(t,y) \otimes \delta_1$$

to obtain a new generalized Markovian family $(\widehat{\mathbb{P}}^{\ddagger}_{t,x})_{t\geq 0,\, x\in E^2\times\{0,1\}}$.

Then let $\mathbb{P}^{\dagger}_{0,(x,x)}$ be the image of $\mathbb{P}^{\ddagger}_{0,(x,x,0)}$ under the natural projection of $\mathbb{M}(\mathbb{R}_+, E^2 \times \{0,1\})$ on $\mathbb{M}(\mathbb{R}_+, E^2)$. It is not difficult to convince oneself that it is indeed a coupling of $\mathbb{P}_{0,x}$ with $\widehat{\mathbb{P}}_{0,x}$.

Furthermore, we have, denoting by $(Z_t)_{t\geq 0}$ the canonical coordinates on $\{0,1\}$,

$$\begin{aligned} \mathbb{P}^{\dagger}_{0,(x,x)}[(X_t)_{0\leq t\leq T} \neq (\widehat{X}_t)_{0\leq t\leq T}] \;&\leq\; \mathbb{P}^{\ddagger}_{0,(x,x,0)}[Z_T = 1] \\ &=\; 1 - \exp(-rT) \end{aligned}$$

because under $\mathbb{P}^{\ddagger}_{0,(x,x,0)}$, Z_T is distributed as $\mathbf{1}_{[0,T]}(S)$, where S is an exponential random variable of parameter r.

■

The latter coupling is rather a crude one: roughly speaking, up to any given time we are considering only two possibilities: either the trajectories of the two processes coincide, either they are different. But we will need to be a little more precise, by quantifying the distance between the positions of the two processes, more specifically in the case of a system of particles, we would like to know how many particles are different. As there is no a priori metric on the state space in our setting, this is the only natural comparison we can consider!

So let us give a general definition of a particle system with interactions changing one particle at each time.

First we still assume that we are given a Markov family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ on E. Then let $N \in \mathbb{N}^*$ be a number of particles. As underlying "unperturbed" Markovian family on E^N, we consider the one, again written $(\mathbb{P}_{t,x})_{t\geq 0, x\in E^N}$, which corresponds to a Markov process on E^N whose coordinates evolve independently and according to $(\mathbb{P}_{t,x})_{t\geq 0, x\in E}$, and which is rigorously defined by

$$\forall\, t \geq 0, \forall\, x = (x_1,\ldots,x_N) \in E^N, \qquad \mathbb{P}_{t,x} = \bigotimes_{1\leq i\leq N} \mathbb{P}_{t,x_i}$$

clearly it also satisfies (H2).

For each $1 \leq i \leq N$, we consider a locally bounded nonnegative kernel $\widehat{R}_i$ from $\mathbb{R}_+ \times E^N$ to E. In order to simplify the presentation, we will work under the hypothesis that the quantity $r \stackrel{\text{def.}}{=} N\widehat{R}_i(t,x,E)$ does not depend on $1 \leq i \leq N$, $t \geq 0$ and $x \in E$.

From these kernels, we define a new one $\widehat{R}$ from $\mathbb{R}_+ \times E^N$ to E^N, via the formulae

$\forall\ t \geq 0,\ \forall\ x = (x_i)_{1\leq i\leq N} \in E^N,$

$$\widehat{R}(t,x) \;=\; \sum_{1\leq i\leq N} \delta_{x_1} \otimes \cdots \otimes \delta_{x_{i-1}} \otimes R_i(t,x) \otimes \delta_{x_{i+1}} \otimes \cdots \otimes \delta_{x_N}$$

Let us denote by $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0, x\in E^N}$ the perturbation of $(\mathbb{P}_{t,x})_{t\geq 0, x\in E^N}$ by this kernel $\widehat{R}$. The mechanism of its interactions at any selected time $t \geq 0$ can be interpreted in the following way: one choose uniformly an indice $1 \leq i \leq N$, and then the coordinate x_i of a position $x \in E^N$ is replaced by the value obtained from a sampling according to the law $\widehat{R}_i(t,x,\cdot)/\widehat{R}_i(t,x,E)$, the other coordinates remain unchanged. Between the random interacting times, the particles evolve independently.

Let us also denote by $\nu^{(r)}$ the law of the usual Poisson process of parameter r on $\mathbb{M}(\mathbb{R}_+,\mathbb{N})$, the set of "càdlàg" paths from $\mathbb{R}_+$ to $\mathbb{N}$. Then we get the following result which in this situation is more precise than the proposition 2.9:

Proposition 2.10 *For any $x \in E^N$, there exists a coupling $\mathbb{P}^\dagger_{0,(x,x)}$ of $\mathbb{P}_{0,x} \otimes \nu^{(r)}$ with $\widehat{\mathbb{P}}_{0,x}$, such that if we denote by $((X^{(i)}_t)_{1\leq i\leq N}, K_t, (\widehat{X}^{(i)}_t)_{1\leq i\leq N})_{t\geq 0}$ the canonical coordinates on $\mathbb{M}(\mathbb{R}_+, E^N \times \mathbb{N} \times E^N)$, then we have*

$$\forall\ T \geq 0, \qquad \mathbb{P}^\dagger_{0,(x,x)}\left[\sum_{1\leq i\leq N} \mathbf{1}_{(X^{(i)}_t)_{0\leq t\leq T} \neq (\widehat{X}^{(i)}_t)_{0\leq t\leq T}} \geq K_T\right] \;=\; 0$$

In particular, we get that

$$\forall\ T \geq 0,\ \forall\ k \geq 0, \qquad \mathbb{P}^\dagger_{0,(x,x)}\left[\sum_{1\leq i\leq N} \mathbf{1}_{(X^{(i)}_t)_{0\leq t\leq T} \neq (\widehat{X}^{(i)}_t)_{0\leq t\leq T}} \geq k\right] \;\leq\; \sum_{l\geq k} \frac{(rT)^l}{l!} \exp(-rT)$$

Here again, the above results have to be interpreted in a special way: let $(F,\mathcal{F})$ be a measurable space, we endow $F^N \times F^N$ with its canonical coordinates $((X_i)_{1\leq i\leq N}, (\widehat{X}_i)_{1\leq i\leq N})$. Let m be a nonnegative finite measure on $(F^N \times F^N, \mathcal{F}^{\otimes N} \otimes \mathcal{F}^{\otimes N})$, for any $0 \leq k \leq N$, we define

$$m\left(\sum_{1\leq i\leq N} \mathbf{1}_{X_i \neq \widehat{X}_i} \geq k\right) \;=\; \sup m_1(F^N \times F^N)$$

where the supremum is taken over all nonnegative measure $m_1 \leq m$ on $(F^N \times F^N, \mathcal{F}^{\otimes N} \otimes \mathcal{F}^{\otimes N})$ which can be decomposed into

$$m_1 \;=\; \sum_{A\subset\{1,\ldots,N\},\, \mathrm{card}(A)=k} m_{1,A}$$

where $m_{1,A}$ satisfies that its image $\widetilde{m}_{1,A}$ by the natural projection from $F^N \times F^N$ to $F^A \times F^A$ verifies $\widetilde{m}_{1,A}(\triangle(F^A)) = m_{1,A}(F^A \times F^A)$ (in this case also the supremum is a maximum, but except for $k = N$, the optimal above decomposition is not unique in general).

By now the signification of the second inequality of the proposition is clear. For the first equality, it means that when, for $k \geq 0$ given, we look at the restriction of $\mathbb{P}^\dagger_{0,(x,x)}$ to the set $\{K_T = k\}$ and consider its projection $\mathbb{P}^{\dagger,k}_{0,(x,x)}$ to $\mathbb{M}([0,T], E^N \times E^N)$, then it satisfies

$$\mathbb{P}^{\dagger,k}_{0,(x,x)}\left[\sum_{1\leq i\leq N} \mathbf{1}_{(X^{(i)}_t)_{0\leq t\leq T} \neq (\widehat{X}^{(i)}_t)_{0\leq t\leq T}} \geq k\right] \;=\; 0$$

We could go further and give a meaning to the affirmation that

$$\mathbb{P}^{\dagger}_{0,(x,x)}\left[\exists\, T\geq 0\,:\,\sum_{1\leq i\leq N}\mathbb{1}_{(X^{(i)}_t)_{0\leq t\leq T}\neq(\widehat{X}^{(i)}_t)_{0\leq t\leq T}}\geq K_T\right] \;=\; 0$$

but we will not need it (be careful, $\sum_{1\leq i\leq N}\mathbb{1}_{(X^{(i)}_t)_{0\leq t\leq T}\neq(\widehat{X}^{(i)}_t)_{0\leq t\leq T}}$ is not a random variable, so one cannot use its monotonicity with respect to $t\geq 0$, rather one has to use a measurable conditioning by $(K_t)_{t\geq 0}$, which can be well-defined here, if one consider only the increasing trajectories of $\mathbb{M}(\mathbb{R}_+,\mathbb{N})$ with jumps of height 1 ...).

Proof:

It is quite similar to the proof of proposition 2.9, we begin by considering a generalized Markov family on $E^{2N}\times\mathcal{P}_N\times\mathbb{N}$, where $\mathcal{P}_N$ is the set of the subsets of $\{1,\ldots,N\}$, defined by

$\forall\, t\geq 0,\ \forall\,(x,y,z,k)\in E^{2N}\times\mathcal{P}_N\times\mathbb{N},$

$$\mathbb{P}^{\ddagger}_{t,(x,y,z,k)} \;=\; \left(\bigotimes_{i\notin z}\mathbb{P}^{\ddagger}_{t,x_i,x_i}\bigotimes_{i\in z}(\mathbb{P}_{t,x_i}\otimes\mathbb{P}_{t,y_i})\right)\otimes\delta_{z_{[t,+\infty[}}\otimes\delta_{k_{[t,+\infty[}}$$

Then we introduce a new nonnegative kernel $\widehat{R}^{\ddagger}$ from $\mathbb{R}_+\times E^{2N}\times\mathcal{P}_N\times\mathbb{N}$ to $E^{2N}\times\mathcal{P}_N\times\mathbb{N}$, by taking, for all $t\geq 0$ and all $(x,y,z,k)\in E^{2N}\times\mathcal{P}_N\times\mathbb{N}$,

$$\widehat{R}^{\ddagger}(t,(x,y,z,k)) \;=\; \sum_{1\leq i\leq N}\delta_x\otimes\widehat{R}(t,y)\otimes\delta_{z\cup\{i\}}\otimes\delta_{k+1}$$

and we perturbe the family $(\mathbb{P}^{\ddagger}_{t,(x,y,z,k)})_{t\geq 0,(x,y,z,k)\in E^{2N}\times\mathcal{P}_N\times\mathbb{N}}$ by this kernel to get a new family $(\widehat{\mathbb{P}}^{\ddagger}_{t,(x,y,z,k)})_{t\geq 0,(x,y,z,k)\in E^{2N}\times\mathcal{P}_N\times\mathbb{N}}$. For $x\in E^N$, let $\mathbb{P}^{\dagger}_{0,(x,x)}$ be the image of $\widehat{\mathbb{P}}^{\ddagger}_{0,(x,x,\emptyset,0)}$ under the projection

$$\mathbb{M}(\mathbb{R}_+,E^{2N}\times\mathcal{P}_N\times\mathbb{N})\ni(\omega_1,\omega_2,\omega_3,\omega_4)\quad\mapsto\quad(\omega_1,\omega_4,\omega_2)\in\mathbb{M}(\mathbb{R}_+,E^N\times\mathbb{N}\times E^N)$$

The affirmations of the proposition now follow quite easily from this construction, for instance under $\widehat{\mathbb{P}}^{\ddagger}_{0,(x,x,\emptyset,0)}$, for $t\geq 0$, K_t ($\geq\mathrm{card}(Z_t)$) counts the number of interaction jump(s) proposed during the time interval $[0,t]$, so $(K_t)_{t\geq 0}$ is distributed as a Poisson process of parameter r.

■

In more general situations, as usual, one has to replace r by $\sup_{0\leq t\leq T,\,x\in E}R(t,x,E)$, if he is only interested in what is happening before time $T\geq 0$.

2.4 The interacting particle system

For any given number of particles $N\in\mathbb{N}^*$, the Markovian process $\xi^{(N)}$ announced in the introduction will here be constructed directly from the family $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$, and not merely defined by a martingale problem, as it was the case in [6]. There are three reasons for this choice: first we believe that it emphasizes the close links between the object under study, $(\eta_t)_{t\geq 0}$, and the approximating scheme $(\xi^{(N)})_{N\geq 1}$, which are both deduced directly from the same basic family $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$, and it gives a way to sample the interacting particle processes in practice, at least under the assumption that one knows how to do it wrt $\mathbb{P}_{t,x}$, for any $t\geq 0$ and $x\in E$. Secondly, the direct construction is nicely adapted to coupling arguments. The last reason is even more technical: if one wants to start from the martingale problems, one will have to consider a priori a set of functions on $[0,T]\times E^N$, for $T\geq 0$ given, which in some sense is a (space) tensorization of $\mathcal{A}_T$ (cf [6]). But in general this domain is too small for our purposes, because it is strictly included in the domain of functions giving rise to natural martingales relatively to $\xi^{(N)}$, and one would have to extend it via some closures. As we will see, it is more convenient to first tensorize the the family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$, to perturbe it in a bounded way, and then to consider the general associated martingale problem.

Quite obviously, we will use the above sections to construct the interactions between the coordinates of $\xi^{(N)}$. The underlying "unperturbed" Markovian family is the one previously defined, $(\mathbb{P}_{t,x})_{t\geq 0, x\in E^N}$, corresponding to independent evolutions of coordinates according to $(\mathbb{P}_{t,x})_{t\geq 0, x\in E}$.

For any given horizon $T\geq 0$, let us denote by $(\mathcal{A}_{T,N},\widetilde{\mathcal{L}}_T^{(N)})$ the generator associated as before (recall that it is contingently multi-valued) to this Markovian family. The third point alluded to in the above introductory paragraph just amounts to the observation that in general, $\mathcal{A}_{T,N}$ is strictly larger than the set of functions $f\,:\,[0,T]\times E^N\rightarrow\mathbb{R}$ which can be written as

$$\forall\; t\in[0,T],\,\forall\; x=(x_1,\cdots,x_N)\in E^N,\qquad f(t,x)\;\;=\;\;\prod_{1\leq i\leq N} f_i(t,x_i)$$

with $f_1,\cdots,f_N\in\mathcal{A}_T$.

In order to define the interaction we want to add to the $\widetilde{\mathcal{L}}_T^{(N)}$, for $T\geq 0$, let us introduce the following notation: for all $1\leq i,j\leq N$ and all $x=(x_1,\cdots,x_N)\in E^N$, $x^{i,j}$ is the element of E^N given by

$$\forall\; 1\leq k\leq N,\qquad x_k^{i,j}\;\;=\;\;\left\{\begin{array}{ll} x_k & ,\text{ if } k\neq i\\ x_j & ,\text{ if } k=i\end{array}\right.$$

Then we consider the locally bounded nonnegative kernel $\widehat{R}$ from $\mathbb{R}_+\times E^N$ to E^N defined by

$$\forall\; t\geq 0,\,\forall\; x=(x_1,\cdots,x_N)\in E^N,\qquad \widehat{R}(t,x)\;\;=\;\;\frac{1}{N}\sum_{1\leq i,j\leq N} U_t(x_j)\delta_{x^{i,j}}$$

and will rather write $\widehat{\mathcal{L}}_T^{(N)}$ for its associated generator on $[0,T]\times E^N$:

$\forall\; f\in\mathcal{B}_{\mathrm{b}}([0,T]\times E^N),\,\forall\;(t,x)\in[0,T]\times E^N,$

$$\widehat{\mathcal{L}}_T^{(N)}(f)(t,x)\;\;=\;\;\frac{1}{N}\sum_{1\leq i,j\leq N} U_t(x_j)(f(t,x^{i,j})-f(t,x))$$

Now we are in position, via the results of section 2.2, to construct the Markovian family $(\widehat{\mathbb{P}}_{t,x})_{t\geq 0, x\in E^N}$ whose associated generators are the $(\mathcal{A}_{T,N},\mathcal{L}_T^{(N)})$, for all $T\geq 0$, where

$$\mathcal{L}_T^{(N)}\;\;=\;\;\widetilde{\mathcal{L}}_T^{(N)}+\widehat{\mathcal{L}}_T^{(N)}$$

It appears that at time $0\leq t\leq T$, the generator $\widehat{\mathcal{L}}_T^{(N)}(t,\,\cdot)$ have a tendency to choose among the coordinates of $x\in E^N$ a x_j with a large $U_t(x_j)$ and to replace an other coordinate by this one. This is a Moran selection step with cost function U_t, and the operator $(\mathcal{A}_{T,N},\mathcal{L}_T^{(N)})$ can be seen as a genetic type generator based on the mutation (or a priori) generator $\widetilde{\mathcal{L}}_T^{(N)}$ which makes the coordinates explore independently the space E.

If η_0 is the initial law which has been seen in the introduction, we are particularly interested in the interacting particle system $(\xi_t^{(N,i)})_{t\geq 0, 1\leq i\leq N}$, whose law is the probability, denoted by $\mathbb{P}$ for simplicity, defined on $\mathbb{M}(\mathbb{R}_+,E^N)$ by

$$\forall\; A\in\mathcal{M}(\mathbb{R}_+,E^N),\qquad \mathbb{P}(A)\;\;=\;\;\int_{E^N}\eta_0^{\otimes N}(dx)\widehat{\mathbb{P}}_{0,x}(A)$$

ie we will assume that initially the coordinates of $\xi_0^{(N)}$ are independent and identically distributed according to η_0, but a careful study of the following proofs would indicate how much this assumption can be weakened.

3 Evolution of the tensorized empirical measures

The purpose of this section is to revisit some weak convergence results given in [6] in order to improve and extend them. In the classical approach (cf for instance [15] or [10]), one deduces the weak propagation from the strong one, but we will proceed in the other way round, getting the strong property in section 4 from a generalization of the weak form presented here. More precisely, our main goal is to show

Theorem 3.1 *For all $T > 0$, $n \in \mathbb{N}^*$, $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T$ and $\varphi \in \mathcal{B}_{\mathrm{b}}(E^n)$, we are assured of the bound*

$$\left|\mathbb{E}[\eta_{t_1}^{(N)} \otimes \cdots \otimes \eta_{t_n}^{(N)}(\varphi)] - \eta_{t_1} \otimes \cdots \otimes \eta_{t_n}(\varphi)\right| \quad \leq \quad \epsilon_T\left(\frac{n^2}{N}\right)\|\varphi\|$$

where the empirical measures of the lhs were defined in (2) and where $\epsilon_T : \mathbb{R}_+ \to [0,2]$ is an increasing function depending on T (through the quantity Tu_T, the following computations will give it explicitly), whose behaviour in zero is given by

$$\lim_{a \to 0_+} \frac{\epsilon_T(a)}{a} \quad = \quad 14 + 28u_T T[1 + \exp(u_T T)]$$

The case $n = 1$ could easily be deduced from the estimations proved in [6], nevertheless, in order to deal with the general situation, we have to develop a new approach, which will also enable us to recover this case $n = 1$, but under the less restrictive hypotheses considered here.

The basic idea is to adopt a "dynamical point of view", in some sense interpreting a quantity closely related to $\eta_{t_1}^{(N)} \otimes \cdots \otimes \eta_{t_n}^{(N)}(\varphi)$ as a terminal value, so that we can find nice martingales to calculate its expectation. Unfortunatly its cautious development is as long as its principle is simple.

For $T > 0$, $N, n \in \mathbb{N}^*$ and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq T$ fixed, let us denote by $\widetilde{\eta}_{t_1,\cdots,t_n}^{(N)}$ the "integrated" law on E^n defined by

$$\forall\, \varphi \in \mathcal{B}_{\mathrm{b}}(E^n), \qquad \widetilde{\eta}_{t_1,\cdots,t_n}^{(N)}(\varphi) \quad = \quad \mathbb{E}[\eta_{t_1}^{(N)} \otimes \cdots \otimes \eta_{t_n}^{(N)}(\varphi)]$$

Then the previous theorem just says that in the total variation sense, we have

$$\left\|\widetilde{\eta}_{t_1,\cdots,t_n}^{(N)} - \eta_{t_1} \otimes \cdots \otimes \eta_{t_n}\right\|_{\mathrm{tv}} \quad \leq \quad \epsilon_T\left(\frac{n^2}{N}\right)$$

As we will explain it latter on, the dependence of the upper bound in n^2/N, with $T > 0$ fixed, implies (through the second coupling presented in section 4.2) the same type of convergence as that obtained by Graham and Méléard [10] for the strong propagation of chaos. But if we were less exacting on this point, it could be possible to give a little more straightforward proof of a weaker upper bound with respect to n.

3.1 Actions of the generators

As we are interested in getting results on empirical measures, we will attempt to understand more particularly the action of the generators $\widetilde{\mathcal{L}}_T^{(N)}$ and $\widehat{\mathcal{L}}_T^{(N)}$, for $T \geq 0$ and $n \in \mathbb{N}^*$, on functions of $\mathcal{A}_{T,N}$ whose dependence on the space parameter goes more or less naturally through the mapping

$$\begin{array}{rcl} m^{(N)} \,:\, E^N & \to & \mathcal{P}(E) \\ x = (x_i)_{1\leq i\leq N} & \mapsto & m^{(N)}(x) = \dfrac{1}{N}\displaystyle\sum_{1\leq i\leq N} \delta_{x_i} \end{array}$$

(it simply means that we consider the particles as indistinguishable).

This is what we have already done in [6] for the case $n = 1$. Here we will have to consider probabilities on E^n, and one could think that the natural object replacing $m^{(N)}(x)$, for $x \in E^N$, is $(m^{(N)}(x))^{\otimes n}$, but it seems that (for $1 \leq n \leq N$) it is preferable to first look at

$$m^{\odot(N,n)}(x) \quad \stackrel{\text{def.}}{=} \quad \frac{1}{N^n} \sum_{(i_1,i_2,\cdots,i_n)\in I(N,n)} \delta_{(x_{i_1},x_{i_2},\cdots,x_{i_n})} \quad \in \quad \mathcal{P}(E^n)$$

where $I(N,n)$ is the set of $(i_1, i_2, \cdots, i_n) \in \{1, \cdots, N\}^n$ such that all i_l and i_k are different for $1 \leq l \neq k \leq n$.

More precisely, we will concentrate our study on mappings of the following form

$$\begin{array}{rcl} F_f \,:\, [0,T] \times E^N & \to & \mathbb{R} \\ (t, x) & \mapsto & m^{\odot(N,n)}(x)[f(t, \cdot)] \end{array}$$

where $T \geq 0$ and $f \in \mathcal{A}_{T,n}$ are fixed. The time dependence appearing above will be important in what follows.

We remark that the definition of $m^{\odot(N,n)}$ and the assumed regularity of f imply that $F_f \in \mathcal{A}_{T,N}$, but note that this would not have been so if we had considered $(m^{(N)})^{\otimes n}$ instead of $m^{\odot(N,n)}$.

Indeed, for $(i_1, i_2, \cdots, i_n) \in I(N, n)$ and $f \in \mathcal{A}_{T,n}$, let us designate by $f^{(i_1,i_2,\cdots,i_n)}$ the function belonging to $\mathcal{A}_{T,N}$ and defined by

$$\forall\ 0 \leq t \leq T,\ \forall\ x = (x_1, \cdots, x_N) \in E^N, \qquad f^{(i_1,i_2,\cdots,i_n)}(t, x) \quad = \quad f(t, x_{i_1}, x_{i_2}, \cdots, x_{i_n})$$

(if $(i_1, i_2, \cdots, i_n) \in \{1, \cdots, N\}^n \setminus I(N, n)$, it is not clear that the above mapping belongs to $\mathcal{A}_{T,N}$), then we have

$$F_f \quad = \quad \frac{1}{N^n} \sum_{(i_1,i_2,\cdots,i_n)\in I(N,n)} f^{(i_1,i_2,\cdots,i_n)} \ \in\ \mathcal{A}_{T,N}$$

Now taking into account the obvious observation from the martingale problems that

$$\widetilde{\mathcal{L}}_T^{(N)}(f^{(i_1,i_2,\cdots,i_n)}) \quad = \quad (\widetilde{\mathcal{L}}_T^{(n)}(f))^{(i_1,i_2,\cdots,i_n)}$$

we get the following result:

Lemma 3.2 *Assume that $1 \leq n \leq N$ and let $f \in \mathcal{A}_{T,n}$. Then the next commutation relation holds*

$$\widetilde{\mathcal{L}}_T^{(N)}(F_f) \quad = \quad F_{\widetilde{\mathcal{L}}_T^{(n)}(f)}$$

In fact this lemma is true for all $n \geq 1$, since for $n > N$, the usual conventions give $m^{\odot n}(x) \equiv 0$.

In order to describe the action of $\widehat{\mathcal{L}}_T^{(N)}$ on functions of type F_f, we take into consideration a renormalisation of the Moran kernel from $\mathbb{R}_+ \times E^n$ to E^n, and more accurately we will need the restriction to $\mathcal{B}_{\mathrm{b}}([0,T] \times E^n)$ of its associated generator, which is just given by

$$l_{T,N,n} \quad \stackrel{\text{def.}}{=} \quad \frac{n}{N}\widehat{\mathcal{L}}_T^{(n)}$$

Its interest comes from the

Lemma 3.3 *For $1 \leq n \leq N$ and $T \geq 0$ fixed, we consider any function $f \in \mathcal{B}_{\mathrm{b}}([0,T \times E^n)$. Then we get that*

$$\widehat{\mathcal{L}}_T^{(N)}(F_f) \quad = \quad F_{\bar{U}^{(n)}f+l_{T,N,n}(f)} - F_{\bar{U}^{(n)}}F_f$$

where $\bar{U}^{(n)}$ stands for the restriction on $[0,T] \times E^n$ of mapping defined by

$$\forall\ t \geq 0, \forall\ y = (y_i)_{1\leq i\leq n} \in E^n, \qquad \bar{U}(n)(t,y) \quad = \quad \sum_{1\leq i\leq n} U(t,y_i)$$

Proof:

This is just basic combinatorial computations: for all $0 \leq t \leq T$ and all $x = (x_i)_{1\leq i\leq N} \in E^N$,

$$\begin{aligned}
&\widehat{\mathcal{L}}_T^{(N)}(F_f)(t,x) \\
&= \frac{1}{N}\sum_{1\leq i,j\leq N}(m^{\odot(N,n)}(x^{i,j})[f_t] - m^{\odot(N,n)}(x)[f_t])U_t(x_j) \\
&= \frac{1}{N^{n+1}}\sum_{1\leq i,j\leq N}\ \sum_{(i_1,\ldots,i_n)\in I(N,n)}(f_t(x_{i_1}^{i,j},\ldots,x_{i_n}^{i,j}) - f_t(x_{i_1},\ldots,x_{i_n}))U_t(x_j) \\
&= \frac{1}{N^{n+1}}\sum_{(i_1,\ldots,i_n)\in I(N,n)}\ \sum_{1\leq k\leq n}\ \sum_{1\leq j\leq N}(f_t(x_{i_1}^{i_k,j},\ldots,x_{i_n}^{i_k,j}) - f_t(x_{i_1},\ldots,x_{i_n}))U_t(x_j) \\
&= \frac{1}{N^{n+1}}\sum_{(i_1,\ldots,i_n)\in I(N,n)}\ \sum_{1\leq k\leq n}\ \sum_{1\leq j\leq N} f_t(x_{i_1},\ldots,x_{i_{k-1}},x_j,x_{i_{k+1}},\ldots,x_{i_n})U_t(x_j) \\
&\quad -nm^{(N)}(x)[U_t]m^{\odot(N,n)}(x)[f_t] \\
&= \frac{N-n+1}{N^{n+1}}\sum_{1\leq k\leq n}\ \sum_{\substack{(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n)\in I(N,n-1) \\ j\in\{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n\}}} f_t(x_{i_1},\ldots,x_{i_{k-1}},x_j,x_{i_{k+1}},\ldots,x_{i_n})U_t(x_j) \\
&\quad +\frac{N-n+1}{N^{n+1}}\sum_{1\leq k\leq n}\ \sum_{\substack{(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n)\in I(N,n-1) \\ j\notin\{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n\}}} f_t(x_{i_1},\ldots,x_{i_{k-1}},x_j,x_{i_{k+1}},\ldots,x_{i_n})U_t(x_j) \\
&\quad -nm^{(N)}(x)[U_t]m^{\odot(N,n)}(x)[f_t] \\
&= \frac{N-n+1}{N^{n+1}}\sum_{1\leq k\leq n}\ \sum_{l\in\{1,\ldots,n\}\setminus\{k\}}\ \sum_{(i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n)\in I(N,n-1)} f_t(x_{i_1},\ldots,x_{i_{k-1}},x_{i_l},x_{i_{k+1}},\ldots,x_{i_n})U_t(x_j) \\
&\quad +\frac{N-n+1}{N^{n+1}}\sum_{(i_1,\ldots,i_n)\in I(N,n)} f_t(x_{i_1},\ldots,x_{i_n})\sum_{1\leq k\leq n}U_t(x_{i_k}) \\
&\quad -nm^{(N)}(x)[U_t]m^{\odot(N,n)}(x)[f_t]
\end{aligned}$$

Note that the intermediate term in the last expression is

$$\frac{N-n+1}{N}m^{\odot(N,n)}(f_t\bar{U}_t^{(n)})$$

so it just remains to treat the first term which we decompose into the two quantities:

$$\begin{aligned}
&\frac{1}{N^{n+1}}\sum_{1\leq k\leq n}\sum_{l\neq k}\sum_{(i_1,\ldots,i_n)\in I(N,n-1)}(f_t(x_{i_1},\ldots,x_{i_{k-1}},x_{i_l},x_{i_{k+1}},\ldots,x_{i_n})-f_t(x_{i_1},\ldots,x_{i_n}))U_t(x_{i_l})\\
&+\frac{1}{N^{n+1}}\sum_{1\leq k\leq n}\sum_{l\neq k}\sum_{(i_1,\ldots,i_n)\in I(N,n-1)}f_t(x_{i_1},\ldots,x_{i_n})U_t(x_{i_l})\\
=\ &\frac{1}{N^n}\sum_{(i_1,\ldots,i_n)\in I(N,n)}l_{T,N,n}(f)(t,x_{i_1},\ldots,x_{i_n})\\
&+\frac{n-1}{N^{n+1}}\sum_{(i_1,\ldots,i_n)\in I(N,n)}f(t,x_{i_1},\ldots,x_{i_n})\bar{U}_t^{(n)}(x_{i_1},\ldots,x_{i_n})\\
=\ &m^{\odot(N,n)}(x)[l_{T,N,n}(f)(t,\cdot)]+\frac{n-1}{N}m^{\odot(N,n)}(x)(f_t\bar{U}_t^{(n)})
\end{aligned}$$

So in the end, we obtain that for all $0\leq t\leq T$ and all $x\in E^N$,

$$\begin{aligned}
\widehat{\mathcal{L}}_T^{(N)}(F_f)(t,x) &= m^{\odot(N,n)}(x)[\bar{U}_t^{(n)}f_t+l_{T,N,n}(f)(t,\cdot)]-nm^{(N)}(x)[U_t]m^{\odot(N,n)}(x)[f_t]\\
&= m^{\odot(N,n)}(x)[\bar{U}_t^{(n)}f_t+l_{T,N,n}(f)(t,\cdot)]-m^{\odot(N,n)}(x)[\bar{U}_t^{(n)}]m^{\odot(N,n)}(x)[f_t]
\end{aligned}$$

which is the announced result.

■

This leads us to consider for $1\leq n\leq N$, $(\mathbb{P}_{t,x}^{\odot(N,n)})_{t\geq 0,x\in E^n}$ the Markovian family on E^n construct as in section 2.4 by perturbing with the bounded operators $l_{T,N,n}$, for $T\geq 0$, the generators of the n-product of independent coordinates evolving according to $(\mathbb{P}_{t,x})_{t\geq 0,x\in E}$. We will also denote by $Y=(Y_t)_{t\geq 0}$ the canonical coordinate process on $\mathbb{M}(\mathbb{R}_+,E^n)$.

Besides, the horizon $T\geq 0$ and a function $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$ being fixed, we introduce the mapping defined on $[0,T]\times E^n$ by

$$\forall\ 0\leq t\leq T,\ \forall\ y\in E^n,\qquad G_{T,\varphi}^{\odot(N,n)}(t,y)\ =\ \mathbb{E}_{t,y}^{\odot(N,n)}\left[\exp\left(\int_t^T\bar{U}^{(n)}(s,Y_s)\,ds\right)\varphi(Y_T)\right]$$

To see its interest, let us furthermore define a process $\Gamma_{T,\varphi}=(\Gamma_{T,\varphi}(t))_{0\leq t\leq T}$ by

$$\forall\ 0\leq t\leq T,\qquad \Gamma_{T,\varphi}(t)\ =\ \exp\left(\int_0^t n\eta_s^{(N)}(U_s)\,ds\right)\eta_t^{\odot(N,n)}(G_{T,\varphi}^{\odot(N,n)}(t,\cdot))$$

where by definition $\eta_t^{\odot(N,n)}=m^{\odot(N,n)}(\xi_t^{(N)})$.

Then we have

Proposition 3.4 *The process $\Gamma_{T,\varphi}$ is a martingale.*

Proof:

We begin by looking at the process given by

$$\forall\ 0\leq t\leq T,\qquad R_t\ \stackrel{\text{def.}}{=}\ \eta_t^{\odot(N,n)}(G_{T,\varphi}^{\odot(N,n)}(t,\cdot))$$

In order to show how it can give rise to a martingale, we have to see that $H_{T,\varphi}$ belongs to $\mathcal{A}_{T,N}$ and to calculate $\mathcal{L}_T^{(N)}(H_{T,\varphi})$, where the mapping $H_{T,\varphi}$ is defined on $[0,T]\times E^N$ by $H_{T,\varphi} = F_{G_{T,\varphi}^{\odot(N,n)}}$.

But according to the corollary 2.7, we know that $G_{T,\varphi}^{\odot(N,n)} \in \mathcal{A}_{T,n}$ and that

$$\forall\ 0\leq t\leq T,\ \forall\ y\in E^n,\qquad \widetilde{\mathcal{L}}_T^{(n)}(G_{T,\varphi}^{\odot(N,n)})(t,y) \ = \ -\bar{U}_t^{(n)}(y)G_{T,\varphi}^{\odot(N,n)}(t,y) - l_{T,N,n}(G_{T,\varphi}^{\odot(N,n)})(t,y)$$

Now taking into account the lemmas 3.2 and 3.3, it appears that

$$\forall\ 0\leq t\leq T,\ \forall\ x\in E^N,\qquad \mathcal{L}_T^{(N)}(H_{T,\varphi})(t,x) \ = \ -nm^{(N)}(x)[U_t]H_{T,\varphi}(t,x)$$

so

$$\left(R_t + n\int_0^t \eta_s^{(N)}[U_s]R_s\,ds\right)_{0\leq t\leq T}$$

is a martingale.

Then the proposition can be deduced without difficulty, via standard manipulations, under the precautions already presented in the proof of lemma 2.1.

■

More generally, the same arguments show that for all $0\leq t\leq T$ and for all $x\in E^N$, the process

$$\left(\exp\left(\int_t^s n\eta_u^{(N)}(U_u)\,du\right)\eta_s^{\odot(N,n)}(G_{T,\varphi}^{\odot(N,n)}(s,\cdot))\right)_{t\leq s\leq T}$$

is a martingale under $\widehat{\mathbb{P}}_{t,x}$, with respect to the usual filtration.

Now let us define for $t\geq 0$, the random measure

$$\gamma_t^{(N)} \ = \ \exp\left(\int_0^t \eta_s^{(N)}(U_s)\,ds\right)\eta_t^{(N)}$$

The previous martingales will enable us to approximate the quantity

$$\mathbb{E}[\gamma_{t_1}^{(N)}\otimes\gamma_{t_2}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\varphi)]$$

where $n\in\mathbb{N}^*$, $0\leq t_1\leq t_2\leq\cdots\leq t_n\leq T$ and $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$.

But once again, we have to introduce a new object, looking like a generalisation/composition of the $G_{T,\varphi}^{\odot(N,n)}$. It is a family of operators, the $K_{t_0,t_1,\ldots,t_n}^{\odot(N,n)}$, indexed by $n\in\mathbb{N}^*$ and $0\leq t_0\leq t_1\leq\cdots\leq t_n$, and acting respectively on the $\mathcal{B}_{\mathrm{b}}(E^n)$. They are defined by induction on $n\in\mathbb{N}^*$:

- When $n=1$, we are only considering a Feynman-Kac semigroup associated to our initial Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,x\in E}$:

$$\forall\ 0\leq t_0\leq t_1,\ \forall\ \varphi\in\mathcal{B}_{\mathrm{b}}(E),\ \forall\ y\in E,\qquad K_{t_0,t_1}^{\odot(N,1)}(\varphi)(y) \ = \ \mathbb{E}_{t_0,y}\left[\exp\left(\int_{t_0}^{t_1} U_s(X_s)\,ds\right)\varphi(X_{t_1})\right]$$

- Then, assuming that all the operators $K_{t_0,t_1,\ldots,t_n}^{\odot(N,n)}$ have been constructed, for a given $n\geq 1$, we define

$$\forall\ 0\leq t_0\leq\cdots\leq t_{n+1},\ \forall\ \varphi\in\mathcal{B}_{\mathrm{b}}(E^{n+1}),\ \forall\ y\in E^{n+1},\quad K_{t_0,t_1,\ldots,t_{n+1}}^{\odot(N,n+1)}(\varphi)(y) \ = \ G_{t_1,\Psi_{t_1,t_2,\ldots,t_{n+1}}}^{\odot(N,n+1)}(t_0,y)$$

where $\Psi_{t_1,t_2,\ldots,t_n}$ is the mapping given by

$$\forall\ z=(z_1,\ldots,z_{n+1})\in E^{n+1},\qquad \Psi_{t_1,t_2,\ldots,t_{n+1}}(z)\ =\ K^{\odot(N,n)}_{t_1,t_2,\ldots,t_{n+1}}(\varphi_{z_1})(z_2,\ldots,z_n)$$

(as we will often use it from now on, let us recall that we are assuming that the following convention is enforced: when some variables are put in the subscript of a function, it means that we are considering the function where these variables are fixed, eg φ_{z_1} is $\varphi(z_1,\cdot)$, for any given $z_1\in E$).

We will give an interpretation of the above operators in next section, nevertheless to justify their study, we begin by presenting why they are natural in our context.

Proposition 3.5 *For all $n\in\mathbb{N}^*$, all $0\le t_1\le\cdots\le t_n$ and all $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$, we have the estimation*

$$\left|\mathbb{E}[\gamma^{(N)}_{t_1}\otimes\cdots\otimes\gamma^{(N)}_{t_n}(\varphi)]-\eta_0^{\otimes n}[K^{\odot(N,n)}_{0,t_1,\ldots,t_n}(\varphi)]\right|\ \le\ \frac{i(N,n)}{1+i(N,n)}\,\|\varphi\|\,\eta_0^{\otimes n}[K^{\odot(N,n)}_{0,t_1,\ldots,t_n}(\mathbb{1})]$$

where

$$i(N,n)\ =\ 1-\prod_{1\le i\le n-1}\left(1-\frac{i}{N}\right)\ =\ \frac{N^n-\mathrm{card}(I(N,n))}{N^n}$$

Proof:

We will look at the lhs as a telescopic sum. The basic computation comes directly from the note after the proposition 3.4, via an application of the Markov property at time t_p, and says that for any $1\le p\le n-1$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int\mathcal{E}^{n-p}_{t_p,t_{p+1}}\eta^{\odot(N,n-p)}_{t_{p+1}}[K^{\odot(N,n-p-1)}_{t_{p+1},\ldots,t_n}(\varphi_{z_1,\ldots,z_p,\cdot})(\cdot)]\,\mathcal{E}^{n-p}_{0,t_p}\gamma^{(N)}_{t_1}(dz_1)\cdots\gamma^{(N)}_{t_p}(dz_p)\right]\\
&=\ \mathbb{E}\left[\int\eta^{\odot(N,n-p)}_{t_p}[G^{\odot(N,n-p)}_{t_{p+1},K^{\odot(N,n-p-1)}_{t_{p+1},\ldots,t_n}(\varphi_{z_1,\ldots,z_p,\cdot})(\cdot)}(t_p,\cdot)]\,\mathcal{E}^{n-p}_{0,t_p}\gamma^{(N)}_{t_1}(dz_1)\cdots\gamma^{(N)}_{t_p}(dz_p)\right]\\
&=\ \mathbb{E}\left[\int\eta^{\odot(N,n-p)}_{t_p}[K^{\odot(N,n-p)}_{t_p,\ldots,t_n}(\varphi_{z_1,\ldots,z_{p-1}})]\,\mathcal{E}^{n-p}_{0,t_p}\gamma^{(N)}_{t_1}(dz_1)\cdots\gamma^{(N)}_{t_p}(dz_p)\right]
\end{aligned}$$

where we have taken the convention that $K^{\odot(N,0)}_{t_n}$ is the identity operator, for any $t_n\ge0$, and where the following notation has been used:

$$\forall\ 0\le s\le t,\qquad \mathcal{E}_{s,t}\ =\ \exp\left(\int_s^t\eta^{(N)}_u(U_u)\,du\right)$$

Let us remark that we can write

$$\eta^{(N)}_{t_p}\otimes\eta^{\odot(N,n-p)}_{t_p}-\eta^{\odot(N,n-p+1)}_{t_p}\ =\ \frac{1}{N^{n-p+1}}\sum_{(i_p,\ldots,i_n)\in I(N,n-p)}\ \sum_{i\in\{i_p,\ldots,i_n\}}\delta_{(\xi^{(N,i)}_{t_p},\xi^{(N,i_p)}_{t_p},\ldots,\xi^{(N,i_n)}_{t_p})}$$

so at any fixed $(z_1,\ldots,z_{p-1})\in E^{p-1}$, we get

$$\begin{aligned}
&\Bigg|\mathbb{E}\left[\int\eta^{\odot(N,n-p)}_{t_p}[K^{\odot(N,n-p)}_{t_p,\ldots,t_n}(\varphi_{z_1,\ldots,z_{p-1}})]\,\mathcal{E}^{n-p}_{0,t_p}\gamma^{(N)}_{t_1}(dz_1)\cdots\gamma^{(N)}_{t_p}(dz_p)\right]\\
&-\mathbb{E}\left[\int\mathcal{E}^{n-p+1}_{t_{p-1},t_p}\eta^{\odot(N,n-p+1)}_{t_p}[K^{\odot(N,n-p)}_{t_p,\ldots,t_n}(\varphi_{z_1,\ldots,z_{p-1},\cdot})(\cdot)]\,\mathcal{E}^{n-p+1}_{0,t_{p-1}}\gamma^{(N)}_{t_1}(dz_1)\cdots\gamma^{(N)}_{t_{p-1}}(dz_{p-1})\right]\Bigg|
\end{aligned}$$

$$= \frac{1}{N}\left|\mathbb{E}\left[\sum_{p\leq k\leq n}\int \eta_{t_p}^{\odot(N,n-p)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\varphi_{z_1,\ldots,z_{p-1},\xi_{t_p}^{(N,i_k)}})]\,\mathcal{E}_{0,t_p}^{n-p+1}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_{p-1})\right]\right|$$

$$\leq \|\varphi\|\,\frac{1}{N}\mathbb{E}\left[\sum_{p\leq k\leq n}\int \eta_{t_p}^{\odot(N,n-p)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\mathbf{1}_{z_1,\ldots,z_{p-1},\xi_{t_p}^{(N,i_k)}})]\,\mathcal{E}_{0,t_p}^{n-p+1}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_{p-1})\right]$$

Now summing these estimations for $1\leq p\leq n-1$, and taking into account that for $p=0$ we also have

$$\mathbb{E}\left[\mathcal{E}_{0,t_1}^{n}\eta_{t_1}^{\odot(N,n)}[K_{t_1,\ldots,t_n}^{\odot(N,n-1)}(\varphi_{\cdot})(\cdot)]\right] = \mathbb{E}\left[\eta_0^{\odot(N,n)}[K_{t_0,\ldots,t_n}^{\odot(N,n)}(\varphi)]\right]$$

we obtain that

$$\left|\mathbb{E}[\gamma_{t_1}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\varphi)]-\mathbb{E}[\eta_0^{\odot(N,n)}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\varphi)]\right|$$

$$\leq \sum_{1\leq p\leq n-1}\left|\mathbb{E}\left[\int \mathcal{E}_{t_p,t_{p+1}}^{n-p}\eta_{t_{p+1}}^{\odot(N,n-p)}[K_{t_{p+1},\ldots,t_n}^{\odot(N,n-p-1)}(\varphi_{z_1,\ldots,z_p,\cdot})(\cdot)]\,\mathcal{E}_{0,t_p}^{n-p}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_p)\right]\right.$$

$$\left.-\mathbb{E}\left[\int \mathcal{E}_{t_{p-1},t_p}^{n-p+1}\eta_{t_p}^{\odot(N,n-p+1)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\varphi_{z_1,\ldots,z_{p-1},\cdot})(\cdot)]\,\mathcal{E}_{0,t_{p-1}}^{n-p+1}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_{p-1}}^{(N)}(dz_{p-1})\right]\right|$$

$$\leq \frac{\|\varphi\|}{N}\sum_{1\leq p\leq n-1}\mathbb{E}\left[\sum_{p\leq k\leq n}\int \eta_{t_p}^{\odot(N,n-p)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\mathbf{1}_{z_1,\ldots,z_{p-1},\xi_{t_p}^{(N,i_k)}})]\,\mathcal{E}_{0,t_p}^{n-p+1}\gamma_{t_1}^{(N)}(dz_1)\cdots\gamma_{t_p}^{(N)}(dz_{p-1})\right]$$

$$= \|\varphi\|\left|\mathbb{E}[\gamma_{t_1}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\mathbf{1})]-\mathbb{E}[\eta_0^{\odot(N,n)}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\mathbf{1})]\right|$$

But let us come back to the above intermediate step in the case $\varphi=\mathbf{1}$. Using the fact that the expression

$$K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\mathbf{1}_{z_1,\ldots,z_{p-1},\xi_{t_p}^{(N,i)}})$$

does not depend on the choice of $1\leq i\leq N$, we realize that

$$\eta_{t_p}^{(N)}\otimes\eta_{t_p}^{\odot(N,n-p)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\mathbf{1}_{z_1,\ldots,z_{p-1},\cdot})(\cdot)] = \left(1-\frac{n-p}{N}\right)\eta_{t_p}^{\odot(N,n-p+1)}[K_{t_p,\ldots,t_n}^{\odot(N,n-p)}(\varphi_{z_1,\ldots,z_{p-1},\cdot})(\cdot)]$$

So considering all the previous steps, we get the equality

$$\mathbb{E}[\gamma_{t_1}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\mathbf{1})] = \prod_{1\leq i\leq n-1}\left(1-\frac{i}{N}\right)\mathbb{E}[\eta_0^{\odot(N,n)}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\mathbf{1})]$$

from which we deduce that

$$\left|\mathbb{E}[\gamma_{t_1}^{(N)}\otimes\cdots\otimes\gamma_{t_n}^{(N)}(\mathbf{1})]-\mathbb{E}[\eta_0^{\odot(N,n)}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\mathbf{1})]\right| \leq i(N,n)\mathbb{E}[\eta_0^{\odot(N,n)}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\mathbf{1})]$$

Then we also notice, due to the initial independence of the particles, that for any $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$,

$$\mathbb{E}[\eta_0^{\odot(N,n)}[\varphi]] = \prod_{1\leq i\leq n}\left(1-\frac{i}{N}\right)\eta_0^{\otimes n}[\varphi]$$

from where follows the result announced in the proposition.

■

Note that classical computations show that

$$i(N,n) \leq \frac{(n-1)^2}{N} \tag{11}$$

The purpose of the next section is to evaluate the operators $K_{0,t_1,\ldots,t_n}^{\odot(N,n)}$, for $n\in\mathbb{N}^*$ and $0\leq t_1\leq\cdots\leq t_n\leq T$.

3.2 Estimates on Moran semigroups

We will use here a preliminary coupling argument to give an upper bound on the difference between $K^{\odot(N,n)}_{0,t_1,\ldots,t_n}$ and $K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}$, for $n\in\mathbb{N}^*$ and $0\leq t_1\leq\cdots\leq t_n\leq T$, where the last operator is constructed in the same way as the former, but assuming that the coordinates evolves independently.

More precisely, for fixed $1\leq n\leq N$, and $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$, we define

$$\forall\, 0\leq t_0\leq t_1,\,\forall\, y\in E^n,\qquad G^{\otimes(N,n)}_{t_1,\varphi}(t_0,y)\ \stackrel{\text{def.}}{=}\ \mathbb{E}_{t,y}\left[\exp\left(\int_t^T \bar{U}^{(n)}(Y_s)\,ds\right)\varphi(Y_T)\right]$$

Then we define the operators $K^{\otimes(N,n)}_{t_0,t_1,\ldots,t_n}$, also indexed by $n\in\mathbb{N}^*$ and $0\leq t_0\leq t_1\leq\cdots\leq t_n$, and acting respectively on the $\mathcal{B}_{\mathrm{b}}(E^n)$, by induction on $n\in\mathbb{N}^*$:

- When $n=1$, we take the sames as before:

$$\forall\, 0\leq t_0\leq t_1,\qquad K^{\otimes(N,1)}_{t_0,t_1}\ =\ K^{\odot(N,1)}_{t_0,t_1}$$

- Next, assuming that all the operators $K^{\otimes(N,n)}_{t_0,t_1,\ldots,t_n}$ have been constructed, for a given $n\geq1$, we define

$$\forall\, 0\leq t_0\leq\cdots\leq t_{n+1},\,\forall\,\varphi\in\mathcal{B}_{\mathrm{b}}(E^{n+1}),\,\forall\, y\in E^{n+1},$$

$$K^{\otimes(N,n+1)}_{t_0,t_1,\ldots,t_{n+1}}(\varphi)(y)\ =\ G^{\otimes(N,n+1)}_{t_1,K^{\otimes(N,n)}_{t_1,t_2\ldots,t_n}(\varphi_\cdot(\cdot))}(t_0,y)$$

In order to take advantage of the considerations of section 2.3, we have to interpret the above operators as something looking as the semigroups associated to some Markovian processes, one being seen as a bounded perturbation of the other.

So let us start with the "tensorized" operators, which will play the role of the "unperturbed" ones. We assume that $n\in\mathbb{N}^*$ and $0\leq t_1\leq\cdots\leq t_{n-1}$ are fixed. We will construct a locally bounded function $V:\mathbb{R}_+\times E^n\to\mathbb{R}_+$ and for any given $y\in E^n$, a probability $\check{\mathbb{P}}^{\otimes(N,n)}_{0,y}$ on $(\mathbb{M}(\mathbb{R}_+,E^n),\mathcal{M}(\mathbb{R}_+,E^n))$ such that for all $t_n\geq t_{n-1}$, all $y\in E^n$ and all $\varphi\in\mathcal{B}_{\mathrm{b}}(E^n)$,

$$K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}(\varphi)(y)\ =\ \check{\mathbb{E}}^{\otimes(N,n)}_{0,y}\left[\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)\varphi(Y_{t_n})\right]\tag{12}$$

(as usual, Y is the canonical process).

The latter probability will in fact be a product probability, each coordinate evolving independently (but not according to the same law): if $y=(y_i)_{1\leq i\leq n}$,

$$\check{\mathbb{P}}^{\otimes(N,n)}_{0,y}\ =\ \bigotimes_{1\leq i\leq n-1}\mathbb{P}^{(t_i)}_{0,y_i}\bigotimes\mathbb{P}_{0,y_n}$$

where for $t\in\mathbb{R}_+$ and $z\in E$, $\mathbb{P}^{(t)}_{0,z}$ is just the image of $\mathbb{P}_{0,z}$ under the mapping $J_t:\mathbb{M}(\mathbb{R}_+,E)\to\mathbb{M}(\mathbb{R}_+,E)$ defined by

$$\forall\,\omega\in\mathbb{M}(\mathbb{R}_+,E),\,\forall\, s\geq0,\qquad X_s(J_t(\omega))\ =\ X_{s\wedge t}(\omega)$$

Clearly, these probabilities can be embedded into a Markovian familly $(\check{\mathbb{P}}^{\otimes(N,n)}_{t,y})_{t\geq0,y\in E^n}$, by taking more generally

$$\forall\, t\geq0,\,\forall\, y=(y_i)_{1\leq i\leq n}\in E^n,\qquad \check{\mathbb{P}}^{\otimes(N,n)}_{t,y}\ =\ \bigotimes_{1\leq i\leq n-1}J_{t\vee t_i}(\mathbb{P}_{t,y_i})\bigotimes\mathbb{P}_{t,y_n}$$

where the J_s, for $s \geq t$, are rather seen as acting on $\mathbb{M}([t,+\infty[,E)$, and note that $J_t(\mathbb{P}_{t,z})$ is the Dirac mass at the trajectory of a nonmoving particle starting from $z \in E$ at time $t \geq 0$.

The definition of V is also very simple:

$\forall\ t \geq 0,\ \forall\ y = (y_i)_{1\leq i\leq n} \in E^n,$

$$V(t,y) \quad = \quad \left\{ \begin{array}{ll} \sum_{i\leq j\leq n} U(t,y_j) & \text{, if there exists } 1 \leq i \leq n-1 \text{ such that } t_{i-1} \leq t < t_i \\ U(t,y_n) & \text{, if } t \geq t_{n-1} \end{array} \right.$$

Immediate computations shows that (12) is fulfilled.

Next, but $n \in \mathbb{N}^*$ and $0 \leq t_1 \leq \cdots \leq t_{n-1}$ still fixed, we want to construct for any given $y \in E^n$, a probability $\check{\mathbb{P}}_{0,y}^{\odot(N,n)}$ on $(\mathbb{M}(\mathbb{R}_+,E^n),\mathcal{M}(\mathbb{R}_+,E^n))$ such that for all $t_n \geq t_{n-1}$, all $y \in E^n$ and all $\varphi \in \mathcal{B}_{\mathrm{b}}(E^n)$,

$$K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\varphi)(y) \quad = \quad \check{\mathbb{E}}_{0,y}^{\odot(N,n)}\left[\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)\varphi(Y_{t_n})\right] \tag{13}$$

and it is possible to do it via the perturbation technics of section 2.2. So we just have to describe the corresponding nonnegative kernel $\widehat{R}$ from $\mathbb{R}_+ \times E^n$ to E^n:

$\forall\ t \geq 0,\ \forall\ y = (y_i)_{1\leq i\leq n} \in E^n,$

$$R(t,x,\cdot) \quad = \quad \left\{ \begin{array}{ll} \frac{1}{N}\sum_{i\leq j\neq k\leq n} U(t,y_k)\delta_{y^{j,k}} & \text{, if there exists } 0 \leq i \leq n-1 \text{ such that } t_{i-1} \leq t < t_i \\ 0 & \text{, if } t \geq t_{n-1} \end{array} \right.$$

Again, direct and not very stimulating computations show that (13) is satisfied, where the Markovian family $(\check{\mathbb{P}}_{t,y}^{\odot(N,n)})_{t\geq 0,y\in E^n}$ is the perturbation of $(\check{\mathbb{P}}_{t,y}^{\otimes(N,n)})_{t\geq 0,y\in E^n}$ by $\widehat{R}$.

Now we are in position to use the results of section 2.3.

Proposition 3.6 *For all $T \geq 0$, all $n \in \mathbb{N}^*$, all $0 \leq t_1 \leq \cdots \leq t_n \leq T$, all $y \in E^n$ and all $\varphi \in \mathcal{B}_{\mathrm{b}}(E^n)$, we are assured of the bound*

$$\left|K_{0,t_1,\ldots,t_n}^{\odot(N,n)}[\varphi](y) - K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}[\varphi](y)\right| \quad \leq \quad \widetilde{\epsilon}_T\left(\frac{(n-1)n}{N}\right)\|\varphi\|\,K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}[\mathbf{1}](y)$$

where for any $a \geq 0$,

$$\widetilde{\epsilon}_T\,(a) \quad = \quad 2(1-\exp[-au_TT]) + \exp([\exp(u_TT)-1]au_TT) - 1$$

which is equivalent to

$$u_TT(1+\exp(u_TT))a$$

for small $a > 0$.

Proof:

As usual, we start by fixing the horizon $T \geq 0$ and we work on the interval $[0,T]$. Let $\mathbb{P}$ be a coupling of $\check{\mathbb{E}}_{0,y}^{\otimes(N,n)} \otimes \nu^{(r)}$ and $\check{\mathbb{E}}_{0,y}^{\odot(N,n)}$ satisfying the property of proposition 2.10, with $r = \frac{(n-1)n}{N}u_T$. Then we can write, with the notations introduced there (but replacing X by Y):

$$\left|K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}[\varphi](y) - K_{0,t_1,\ldots,t_n}^{\odot(N,n)}[\varphi](y)\right|$$

$$
\begin{aligned}
&= \left|\mathbb{E}\left[\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)\varphi(Y_{t_n}) - \exp\left(\int_0^{t_n} V(s,\widehat{Y}_s)\,ds\right)\varphi(\widehat{Y}_{t_n})\right]\right| \\
&= \left|\sum_{k\geq 1}\mathbb{E}\left[\left(\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)\varphi(Y_{t_n}) - \exp\left(\int_0^{t_n} V(s,\widehat{Y}_s)\,ds\right)\varphi(\widehat{Y}_{t_n})\right)\mathbb{1}_{\{K_T=k\}}\right]\right| \\
&\leq \left|\sum_{k\geq 1}\mathbb{E}\left[\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)\left(\varphi(Y_{t_n}) - \varphi(\widehat{Y}_{t_n})\right)\mathbb{1}_{\{K_T=k\}}\right]\right| \\
&\quad + \left|\sum_{k\geq 1}\mathbb{E}\left[\left(\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right) - \exp\left(\int_0^{t_n} V(s,\widehat{Y}_s)\,ds\right)\right)\varphi(\widehat{Y}_{t_n})\mathbb{1}_{\{K_T=k\}}\right]\right| \\
&\leq 2\,\|\varphi\|\,\mathbb{E}\left[\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)\mathbb{1}_{\{K_T\geq 1\}}\right] \\
&\quad + \|\varphi\|\sum_{k\geq 1}\mathbb{E}\left[\left|\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right) - \exp\left(\int_0^{t_n} V(s,\widehat{Y}_s)\,ds\right)\right|\mathbb{1}_{\{K_T=k\}}\right] \\
&\leq 2\,\|\varphi\|\,\mathbb{E}\left[\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)\right]\mathbb{P}[K_T\geq 1] \\
&\quad + \|\varphi\|\sum_{k\geq 1}\mathbb{E}\left[\exp\left(\int_0^{t_n} V(s,Y_s)\,ds\right)(\exp(kTu_T)-1)\mathbb{1}_{\{K_T=k\}}\right] \\
&\leq 2\,\|\varphi\|\,K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}[\mathbb{1}](y)(1-\exp(-rT)) \\
&\quad + \|\varphi\|\,K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}[\mathbb{1}](y)\sum_{k\geq 1}(\exp(kTu_T)-1)\frac{(rT)^k}{k!}\exp(-rT) \\
&\leq \epsilon_T\left(\frac{(n-1)n}{N}\right)\|\varphi\|\,K^{\otimes(N,n)}_{0,t_1,\ldots,t_n}[\mathbb{1}](y)
\end{aligned}
$$

where we have used that on the set $\{K_T = k\}$, for $k \geq 0$, we are assured of the bound

$$\int_0^{t_n}\left|V(s,Y_s) - V(s,\widehat{Y}_s)\right|\,ds \leq Tku_T$$

(in the sense described in section 2.3).

■

The above dependence in T is not very nice, this comes from the fact that for large T, K_T is not a good bound on the number of coordinates which are different (which should be bounded by n !). A more cautious analysis would improve this point.

3.3 Proof of theorem 3.1

We begin by explaining why $\gamma_t^{(N)}$, for $t \geq 0$, can be an interesting object to evaluate: mainly because it should be (and is) an approximation of

$$\gamma_t \overset{\text{def.}}{=} \exp\left(\int_0^t \eta_s(U_s)\,ds\right)\eta_s$$

But using the well-known fact (cf. for instance [6], and there is no problem in verifying that it is also true in our new context) that

$$\mathbb{E}_{\eta_0}\left[\exp\left(\int_0^t U_s(X_s)\,ds\right)\right] = \exp\left(\int_0^t \eta_s(U_s)\,ds\right)$$

it appears that for all $\varphi \in \mathcal{B}_{\mathrm{b}}(E)$,

$$\gamma_t(\varphi) \;=\; \mathbb{E}_{\eta_0}\left[\exp\left(\int_0^t U_s(X_s)\,ds\right)\varphi(X_t)\right]$$

which is a quantity linear in $\gamma_0 = \eta_0$, more precisely, the deterministic measure-valued flow $(\gamma_t)_{t\geq 0}$ is obtained from η_0 by the application of the semigroup $K^{\otimes(N,1)}$:

$$\forall\ t \geq 0, \qquad \gamma_t \;=\; \eta_0 K_{0,t}^{\otimes(N,1)} \tag{14}$$

(in the sense of the traditionnal action of kernels on measures).

It was these simple acknowledgements which lead us to believe that the γ_t should be easy to compare with the $\gamma_t^{(N)}$, for $t \geq 0$, and in fact the latters are estimations without biais of the formers (cf. [6], or the proposition 3.5 with $n = 1$). For the higher tensor products ($n \geq 2$) this property is lost (it was foreseeable, because for instance the second order tensors are related to square mean errors bounds), nevertheless the previous computations enable to bound the error:

Proposition 3.7 *For all $n \in \mathbb{N}^*$, all $0 \leq t_1 \leq \cdots \leq t_n$ and all $\varphi \in \mathcal{B}_{\mathrm{b}}(E^n)$, we have the following bound on the biais:*

$$\left|\mathbb{E}[\gamma_{t_1}^{(N)} \otimes \cdots \otimes \gamma_{t_n}^{(N)}(\varphi)] - \gamma_{t_1} \otimes \cdots \otimes \gamma_{t_n}(\varphi)\right| \;\leq\; \widehat{\epsilon}_T\left(\frac{n(n-1)}{N}\right)\|\varphi\|\,\gamma_{t_1} \otimes \cdots \otimes \gamma_{t_n}(\mathbf{1})$$

where

$$\forall\ a \geq 0, \qquad \widehat{\epsilon}_T(a) \;=\; 2\widetilde{\epsilon}_T(a) + a$$

Proof:

As an immediate consequence of the propositions 3.5 and 3.6 and of the upper bound (11), we obtain,

$$\begin{aligned}
&\left|\mathbb{E}[\gamma_{t_1}^{(N)} \otimes \cdots \otimes \gamma_{t_n}^{(N)}(\varphi)] - \eta_0^{\otimes n}[K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}(\varphi)]\right| \\
&\leq\; \left|\eta_0^{\otimes n}[K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}(\varphi) - K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\varphi)]\right| + i(N,n)\,\|\varphi\|\,\eta_0^{\otimes n}[K_{0,t_1,\ldots,t_n}^{\odot(N,n)}(\mathbf{1})] \\
&\leq\; (1+i(N,n))\widetilde{\epsilon}_T\left(\frac{(n-1)n}{N}\right)\|\varphi\|\,\eta_0^{\otimes n}[K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}(\mathbf{1})] + i(N,n)\,\|\varphi\|\,\eta_0^{\otimes n}[K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}(\mathbf{1})] \\
&\leq\; \widehat{\epsilon}_T\left(\frac{n(n-1)}{N}\right)\|\varphi\|\,\eta_0^{\otimes n}[K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}(\mathbf{1})]
\end{aligned}$$

and the result follows from the equality

$$\eta_0^{\otimes n}[K_{0,t_1,\ldots,t_n}^{\otimes(N,n)}(\varphi)] \;=\; \gamma_{t_1} \otimes \cdots \otimes \gamma_{t_n}(\varphi)$$

which in turn comes from (14) and the product structure.

■

The above approximation has the interesting property to be "self-improving":

Proposition 3.8 *For all $n \in \mathbb{N}^*$, all $0 \leq t_1 \leq \cdots \leq t_n$ and all $\varphi \in \mathcal{B}_{\mathrm{b}}(E^n)$, we have the following bound on the square mean error:*

$$\left|\mathbb{E}[(\gamma_{t_1}^{(N)} \otimes \cdots \otimes \gamma_{t_n}^{(N)}(\varphi) - \gamma_{t_1} \otimes \cdots \otimes \gamma_{t_n}(\varphi))^2]\right| \;\leq\; \bar{\epsilon}_T\left(\frac{n^2}{N}\right)\|\varphi\|^2\,(\gamma_{t_1} \otimes \cdots \otimes \gamma_{t_n}(\mathbf{1}))^2$$

where

$$\forall\ a \geq 0, \qquad \bar{\epsilon}_T(a) \;=\; \widehat{\epsilon}_T(4a) + 2\widehat{\epsilon}_T(a)$$

Proof:

In order to simplify the notations, let us write

$$\begin{array}{rcl} \gamma^{(N)\otimes}_{t_1,\ldots,t_n} & = & \gamma^{(N)}_{t_1}\otimes\cdots\otimes\gamma^{(N)}_{t_n} \\ \gamma^{\otimes}_{t_1,\ldots,t_n} & = & \gamma_{t_1}\otimes\cdots\otimes\gamma_{t_n} \end{array}$$

Then we consider the expansion, for $\varphi\in\mathcal{B}_{\rm b}(E^n)$,

$$\begin{array}{l} \mathbb{E}[(\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\varphi)-\gamma^{\otimes}_{t_1,\ldots,t_n}(\varphi))^2] \\ \quad = \quad \mathbb{E}[(\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\varphi))^2]-(\gamma^{\otimes}_{t_1,\ldots,t_n}(\varphi))^2-2\gamma^{\otimes}_{t_1,\ldots,t_n}(\varphi)(\mathbb{E}[\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\varphi)]-\gamma^{\otimes}_{t_1,\ldots,t_n}(\varphi)) \\ \quad \leq \quad \left|\mathbb{E}[\gamma^{(N)\otimes}_{t_1,t_1,t_2,\ldots,t_n}(\varphi\otimes\varphi)]-\gamma^{\otimes}_{t_1,t_1,t_2,\ldots,t_n}(\varphi\otimes\varphi)\right|+2\left\|\varphi\right\|\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbb{1})\left|\mathbb{E}[\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\varphi)]-\gamma^{\otimes}_{t_1,\ldots,t_n}(\varphi)\right| \\ \quad \leq \quad \widehat{\epsilon}_T\left(\frac{4n^2}{N}\right)\left\|\varphi\otimes\varphi\right\|\gamma^{\otimes}_{t_1,t_1,t_2,\ldots,t_n}(\mathbb{1})+2\left\|\varphi\right\|\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbb{1})\widehat{\epsilon}_T\left(\frac{n^2}{N}\right)\left\|\varphi\right\|\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbb{1}) \\ \quad = \quad \bar{\epsilon}_T\left(\frac{n^2}{N}\right)\left\|\varphi\right\|^2(\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbb{1}))^2 \end{array}$$

■

More generally, one can found in the same way explicit bounds of any given moment of an integer order $p\geq 1$ (which will always be asymptotically equivalent to a factor times n^2/N, when this quantity is small, as $p\geq 1$ and $T>0$ are fixed).

The above proposition also emphatizes the basic principle underlying this article: usually in order to study martingales associated to Markov processes, one looks at their increasing processes, which are given by the integration along the trajectories of the famous carrés du champs. But that approach leads to difficulties relative to domains of pregenerators which should be algebras (cf. for instance [6]). Here in order to avoid these kinds of embarrassing problems, in some sense we have straightly worked with the squares of the martingales: they are related to the squares of the functionals we are interested in and since the latters are empirical probabilities acting on some mappings, their squares can be seen as 2-tensorized empirical measures applied on 2-tensorized functions, which we study directly (or at least their closely related $\odot$-product).

Now the proof of theorem 3.1 is quite a standard task: first we write that

$$\begin{array}{l} \eta^{(N)}_{t_1}\otimes\cdots\otimes\eta^{(N)}_{t_n}(\varphi)-\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi) \\ \quad = \quad \frac{1}{\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbb{1})}\left[\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\varphi)-\gamma^{\otimes}_{t_1,\ldots,t_n}(\varphi)+\eta^{(N)}_{t_1}\otimes\cdots\otimes\eta^{(N)}_{t_n}(\varphi)(\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbb{1})-\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\mathbb{1}))\right] \end{array} \tag{15}$$

This enables us to get a preliminary bound on the second moment:

$$\begin{array}{l} \mathbb{E}[(\eta^{(N)}_{t_1}\otimes\cdots\otimes\eta^{(N)}_{t_n}(\varphi)-\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi))^2] \\ \quad \leq \quad 2\frac{1}{(\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbb{1}))^2}\left(\mathbb{E}[(\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\varphi)-\gamma^{\otimes}_{t_1,\ldots,t_n}(\varphi))^2]+\left\|\varphi\right\|^2\mathbb{E}[(\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbb{1})-\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\mathbb{1}))^2]\right) \\ \quad \leq \quad 3\bar{\epsilon}_T\left(\frac{n^2}{N}\right)\left\|\varphi\right\|^2 \end{array}$$

and to conclude we integrate again (15):

$$\mathbb{E}[\eta^{(N)}_{t_1}\otimes\cdots\otimes\eta^{(N)}_{t_n}(\varphi)-\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi)]$$

$$
\begin{aligned}
&= \frac{1}{\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})}\Big[\mathbb{E}[\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\varphi)-\gamma^{\otimes}_{t_1,\ldots,t_n}(\varphi)]+\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi)\mathbb{E}[\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})-\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\mathbf{1})] \\
&\quad +\mathbb{E}[(\eta^{(N)}_{t_1}\otimes\cdots\otimes\eta^{(N)}_{t_n}(\varphi)-\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi))(\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})-\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\mathbf{1}))]\Big] \\
&\leq \frac{1}{\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})}\Big[\mathbb{E}[\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\varphi)-\gamma^{\otimes}_{t_1,\ldots,t_n}(\varphi)]+\|\varphi\|\left|\mathbb{E}[\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})-\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\mathbf{1})]\right| \\
&\quad +\sqrt{\mathbb{E}[(\eta^{(N)}_{t_1}\otimes\cdots\otimes\eta^{(N)}_{t_n}(\varphi)-\eta_{t_1}\otimes\cdots\otimes\eta_{t_n}(\varphi))^2]}\sqrt{\mathbb{E}[(\gamma^{\otimes}_{t_1,\ldots,t_n}(\mathbf{1})-\gamma^{(N)\otimes}_{t_1,\ldots,t_n}(\mathbf{1}))^2]} \\
&\leq \epsilon_T\left(\frac{n^2}{N}\right)\|\varphi\|
\end{aligned}
$$

with

$$\forall\, a\geq 0,\qquad \epsilon_T(a) \;=\; 2[(\widehat{\epsilon}_T(a)+\bar{\epsilon}_T(a))\wedge 1]$$

4 Quantitative strong propagation of chaos

It is time now to present the strong propagation of chaos which was the pretext for all the above "preliminaries". The purpose of this result is to explain the behaviour of the n^{th} first coordinates of the interacting particle system $\xi^{(N)}$ as n^2/N is small and in particular to show they are asymptotically independent (this property accounts for the name "propagation of chaos" due to Kac [11], see for instance [15] in a different context: if initially the coordinates are independent, then in the limit of a large number of particles, any fixed finite number of them end up to be still independent over bounded time interval, despite the interactions).

So let the horizon $T\geq 0$ and the numbers of particles $1\leq n\leq N$ be fixed, the object under study is $\mathbb{P}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]}$ the law of $(\xi^{(N,i)}_t)_{1\leq i\leq n,\,0\leq t\leq T}$ under $\mathbb{P}$. In order to describe its limit, we need more notations.

Recall that η_0 being supposed given, we have at our disposal a flow $(\eta_t)_{t\geq 0}$ of probabilities defined by (1). Starting from them, we introduce the non-negative kernel $\bar{R}$ from $\mathbb{R}_+\times E$ to E given by

$$\forall\, t\geq 0, \forall\, x\in E, \forall\, A\in\mathcal{E},\qquad \bar{R}(t,x,A) \;=\; \int_A U_t(y)\,\eta_t(dy)$$

(informally speaking, an important step in the direction of the whole generality of Graham and Méléard's setting [10] would be to let the rhs depend on $x\in E$ via U_t).

Then we consider the time-inhomogeneous Markovian family $(\bar{\mathbb{P}}_{t,x})_{t\geq 0,\,x\in E}$ which is the perturbation of $(\mathbb{P}_{t,x})_{t\geq 0,\,x\in E}$ by $\bar{R}$. Let $\bar{X}\stackrel{\text{def.}}{=}(\bar{X}_t)_{t\geq 0}$ denote the canonical coordinate process on $\mathbb{M}(\mathbb{R}_+,\mathbb{E})$ under the law

$$\bar{\mathbb{P}}_{\eta_0} \;\stackrel{\text{def.}}{=}\; \int \eta_0(dx)\mathbb{P}_{0,x}$$

(for $T\geq 0$, we will also write $\bar{\mathbb{P}}_{\eta_0,[0,T]}$ for the law of $(\bar{X}_t)_{0\leq t\leq T}$ on $\mathbb{M}([0,T],E)$), the initial law η_0 being always the same one considered everywhere.

Now we have to justify the "nonlinear" aspect of this process we have alluded to in the introduction.

Proposition 4.1 *For any $T\geq 0$, the law of $\bar{X}_T$ under $\bar{\mathbb{P}}_{\eta_0}$ is η_T.*

Proof:

We have to verify that for any fixed horizon $T \geq 0$ and function $\varphi \in \mathcal{B}_{\mathrm{b}}(E)$,

$$\mathbb{E}[\varphi(\bar{X}_T)] \;=\; \eta_T(\varphi) \stackrel{\text{def.}}{=} \mathbb{E}\left[\varphi(X_T)\exp\left(\int_0^T U_s(X_s) - \eta_s(U_s)\,ds\right)\right]$$

So naturally we consider the mapping defined by

$$\begin{array}{rcl} F \;:\; [0,T]\times E & \rightarrow & \mathbb{R} \\ (t,x) & \mapsto & \mathbb{E}_{t,x}\left[\varphi(X_T)\exp\left(\int_t^T U_s(X_s) - \eta_s(U_s)\,ds\right)\right] \end{array}$$

According to corollary 2.7, the time-space generator of $(\bar{X}_t)_{0\leq t\leq T}$ is given on this function by

$$\begin{aligned} \forall\, 0\leq t\leq T, \forall\, x\in E, \qquad \bar{L}_T(F)(t,x) &= L_T(F)(t,x) + \int (F(t,y)-F(t,x))U(t,y)\,\eta_t(dy) \\ &= -U(t,x)F(t,x) + \int F(t,y)U(t,y)\,\eta_t(dy) \end{aligned}$$

from where we get that

$$\mathbb{E}[F(T,\bar{X}_T)] \;=\; \mathbb{E}[F(0,\bar{X}_0)] - \int_0^T \mathbb{E}[U(t,\bar{X}_t)F(t,\bar{X}_t)] - \eta_t(U_tF_t)\,dt$$

which can be expressed as

$$m_T(\varphi) \;=\; \eta_T(\varphi) - \int_0^T m_t(U_tF_t) - \eta_t(U_tF_t)\,dt \tag{16}$$

where m_t is the law of $\bar{X}_t$, for any $t \geq 0$.

This easily implies that

$$\|m_T - \eta_T\|_{\mathrm{tv}} \;\leq\; u_T T\exp(u_T T)\sup_{0\leq t\leq T}\|m_t-\eta_t\|_{\mathrm{tv}}$$

and more generally in the same way we obtain

$$\forall\, 0\leq t\leq T, \qquad \|m_t-\eta_t\|_{\mathrm{tv}} \;\leq\; u_T t\exp(u_T t)\sup_{0\leq s\leq t}\|m_s-\eta_s\|_{\mathrm{tv}}$$

So if $t_0 > 0$ is such that $u_T t_0\exp(u_T t_0) = 1/2$, it appears that $m_t = \eta_t$ for all $0 \leq t \leq t_0$.

Now rather considering $(\bar{X}_{t-t_0})_{t\geq t_0}$ and replacing η_0 by η_{t_0}, we obtain that for all $t_0 \leq t \leq 2t_0$, $m_t = \eta_t$. Thus in a finite number of steps, we can conclude that $\eta_T = m_T$.

■

Note that from (16) we cannot deduce that

$$\|m_T-\eta_T\|_{\mathrm{tv}} \;\leq\; \int_0^T \|m_t-\eta_t\|_{\mathrm{tv}}\,dt$$

just because we have no measurability results for $[0,T] \ni t \mapsto \|m_t-\eta_t\|_{\mathrm{tv}}$.

But the main interest of $\bar{X}$ is that the strong propagation of chaos can be expressed as

Theorem 4.2 *Let $C_T = 4(\exp(u_T T) - 1) + (14 + 28u_T T[1 + \exp(u_T T)])u_T T(u_T T + 1)$, then we are assured of*

$$\limsup_{n^2/N\to 0} \frac{N}{n^2} \left\| \mathbb{P}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]} - \bar{\mathbb{P}}^{\otimes n}_{\eta_0,[0,T]} \right\|_{\mathrm{tv}} \leq C_T$$

So if up to time $T \geq 0$, we are considering only n particles, with $n \ll \sqrt{N}$, then they are asymptotically independent and distributed according to $\bar{\mathbb{P}}_{\eta_0,[0,T]}$.

The proof is based on the next two direct coupling arguments, the crucial ingredient being the theorem 3.1.

Nevertheless, let us mentioned that the dependence of the constant C_T in $T \geq 0$ is very bad, except for the small ones, and we are wondering if it would not be possible to improve it by using this behaviour for small $T > 0$.

4.1 A first coupling

We will present in this section another very simple interacting system on E^N, whose n^{th} first coordinates have a special behaviour (they take information from the other particles but do not have influence on them, so globally the system will no longer be exchangeable) but are close enough to the n^{th} first particles of our previous algorithm (at least for n^2/N small).

So we begin by describing this auxilliary interacting particle system which is also of the general type considered in section 2.3: more precisely, with the usual notations, we perturbe the Markovian family $(\mathbb{P}_{t,x})_{t\geq 0, x\in E^N}$ by the kernel $\check{R}^{(N)}$ defined by

$$\forall\, t \geq 0,\, \forall\, x = (x_i)_{1\leq i\leq N} \in E^N, \qquad \check{R}^{(N)}(x) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=n+1}^{N} U(t, x_j)\delta_{x^{i,j}}$$

where $1 \leq n \leq N$ are fixed, to get a new Markovian family $(\check{\mathbb{P}}_{t,x})_{t\geq 0,\, x\in E^N}$. So for any $T \geq 0$, its associated generator can be written $(\mathcal{A}_{T,N}, \check{\mathcal{L}}_T)$, where

$$\check{\mathcal{L}}^{(N)}_T = \widetilde{\mathcal{L}}^{(N)}_T + \check{\mathcal{L}}^{(N)}_T$$

with the selection generator given by

$\forall\, 0 \leq t \leq T$, $\forall \phi \in \mathcal{B}_{\mathrm{b}}(E^N)$, $\forall\, x = (x_1, \ldots, x_N) \in E^N$,

$$\check{\mathcal{L}}^{(N)}_T(\phi)(t, x) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=2}^{N} \left(\phi(x^{i,j}) - \phi(x)\right) U_t(x_j)$$

In order to avoid confusion with the canonical process on $\mathbb{M}(\mathbb{R}_+, E^N)$, we will denote by $(\xi^{(N,i)}_t)_{1\leq i\leq n,\, t\geq 0}$ and $(\check{\xi}^{(N,i)}_t)_{1\leq i\leq n,\, t\geq 0}$ the processes appearing in the "explicit" construction of the families $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E^N}$ and $(\check{\mathbb{P}}_{t,x})_{t\geq 0,\, x\in E^N}$.

The generator $\check{\mathcal{L}}^{(N)}_T$ is quite similar to the selection operator $\widehat{\mathcal{L}}^{(N)}_T$, except that the particles $\check{\xi}^{(N,i)}_t$, for $n \leq i \leq N$ and $0 \leq t \leq T$, are not permitted to inherite by a selection step the values of $\check{\xi}^{(N,j)}_t$, for $1 \leq j \leq n$. Furthermore, the restriction of $\check{\mathcal{L}}^{(N)}_T$ on functions depending only on the coordinates whose indices belong to $\{n+1, \ldots, N\}$ is equal to $\widehat{\mathcal{L}}^{(N-n)}_T$, up to a factor $(N-n)/N$ and to the reindexing of these indices obtained by adding n.

Note that if we perturbe $(\breve{\mathbb{P}}_{t,x})_{t\geq 0,\, x\in E^N}$ by the kernel given at time $t \geq 0$ and point $x = (x_i)_{1\leq i\leq N} \in E^N$ by

$$\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{n} U(t,x_j)\delta_{x^{i,j}}$$

then we end up with the Markovian family of the algorithm we considered before (due to the unicity of the associated generators). We will use this important feature to show our main result here:

Proposition 4.3 *Let $\breve{\mathbb{P}}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]}$ designate the law of $(\breve{\xi}^{(N,i)}_t)_{1\leq i\leq n,\, 0\leq t\leq T}$ "under" $\int \eta_0^{\otimes N}(dx)\, \breve{\mathbb{P}}_{0,x}$. Then we are assured of*

$$\left\|\mathbb{P}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]} - \breve{\mathbb{P}}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]}\right\|_{\mathrm{tv}} \;\leq\; 4\frac{n^2}{N}(\exp(u_T T)-1)$$

Proof:

As we will use a coupling argument, let us come back to the construction of $(\xi^{(N)}_t)_{0\leq t\leq T}$ (the horizon $T\geq 0$ is assumed to be fixed) which follows from the considerations of section 2.2 and 2.4: we denote by $(S_p)_{p\geq 1}$ the proposed selection times (such that the differences $(S_p - S_{p-1})_{p\geq 1}$ are independent and identically distributed according to exponential laws of parameter Nu_T, with the convention that $S_0 = 0$) and by $(Z_t)_{t\geq 0}$ the corresponding Poisson process. Let us also consider the following independent objects: $(I_p, J_p)_{p\geq 1}$ a family of independent uniformly distributed random variables in $\{1,\cdots,N\}^2$ and $(V_p)_{p\geq 1}$ a family of independent uniformly distributed random variables in $[0,1]$. We can assume that for any $p\geq 1$, the sampling of $\xi^{(N)}_{S_p}$ knowing "$\xi^{(N)}_{S_p-}$" is done according to the next mechanism: we replace the I_p-th coordinate of $\xi^{(N)}_{S_p-}$ by its J_p-th coordinate, if $V_p \leq U_{S_p}(\xi^{(N,J_p)}_{S_p})/u_T$, otherwise we take $\xi^{(N)}_{S_p} = \xi^{(N)}_{T_p-}$ (classical acceptation/rejection procedure).

Meanwhile, from the sequence $(I_p, J_p)_{p\geq 1}$ we can define a family $(A_p)_{p\geq 1}$ of random variables taking values in the subsets of $\{1,\ldots,N\}$: we start with $A_0 = \{1,\ldots,n\}$ and if A_p has been defined, we put

$$A_{p+1} \;=\; \begin{cases} A_p \cup \{I_{p+1}\} & \text{, if } J_{p+1}\in A_p \\ A_p & \text{, otherwise} \end{cases}$$

To see its interest, let us remark that using the same intuitive ideas and technical precautions as those presented in section 2.3, we can construct a process $(\breve{\xi}^{(N)}_t)_{0\leq t\leq T}$ whose law will be the restriction to $\mathbb{M}([0,T],E^N)$ of $\int \eta_0^{\otimes N}(dx)\breve{\mathbb{P}}_{0,x}$ and which is coupled to $(\xi^{(N)}_t)_{0\leq t\leq T}$ in the sense that the next property is satisfied:

$$\forall\, p\geq 1,\, \forall\, T\wedge S_p \leq t < T\wedge S_{p+1},\, \forall\, n+1\leq i\leq N, \qquad \breve{\xi}^{(N,i)}_t \neq \xi^{(N,i)}_t \;\Longrightarrow\; i\in A_p$$

(in particular we are starting with $\breve{\xi}^{(N)}_0 = \xi^{(N)}_0$). Heuristically, for $p\geq 0$, $A_p\setminus\{1,\ldots,n\}$ is the set of subscripts $n+1\leq i\leq N$ such that $\breve{\xi}^{(N,i)}_{S_p}$ has a good chance to be different from $\xi^{(N,i)}_{S_p}$.Then it appears that on the set

$$\{Z_T = 0\}\sqcup \bigsqcup_{p\geq 1}\{Z_T = p,\, \forall\, 1\leq q\leq p,\, I_q\not\in\{1,\ldots,n\} \text{ or } J_q\not\in A_p\}$$

we are assured that $(\breve{\xi}^{(N,i)}_t)_{1\leq i\leq n,\, 0\leq t\leq T} = (\xi^{(N,i)}_t)_{1\leq i\leq n,\, 0\leq t\leq T}$.

So with the usual conventions inforced, we get that

$$\begin{aligned}
\mathbb{P}[(\breve{\xi}_t^{(N)})_{0\le t\le T} \neq (\xi_t^{(N)})_{0\le t\le T}] &\le \sum_{p\ge 1} \mathbb{P}[Z_T = p,\, \exists\, 1\le q\le p,\, I_q\in\{1,\ldots,n\} \text{ and } J_q \in A_q] \\
&= \sum_{p\ge 1} \exp(Nu_T T)\frac{(Nu_T T)^p}{p!}\frac{n}{N^2}\sum_{q=1}^{p}\mathbb{E}[\mathrm{card}(A_q)]
\end{aligned}$$

(here $\mathbb{P}$ denotes the underlying probability and not the law of the interacting particle system).

This leads us to consider the sequence $(B_p)_{p\ge 0} \stackrel{\text{def.}}{=} (\mathrm{card}(A_p))_{p\ge 0}$. It is quite clear that it is an increasing inhomogeneous Markov chain taking values in $\{n,\ldots,N\}$, whose probabilities of transition are given by

$$\forall\, p\ge 0, \forall\, n\le k, l\le N, \qquad \mathbb{P}[B_{p+1} = l | B_p = k] \;=\; \begin{cases} \frac{(N-k)k}{N^2} & ,\ \text{if } l = k+1 \\ 1-\frac{(N-k)k}{N^2} & ,\ \text{if } l = k \\ 0 & ,\ \text{otherwise} \end{cases}$$

So we get that for $p\ge 0$,

$$\begin{aligned}
\mathbb{E}[B_{p+1}] &= \mathbb{E}[\mathbb{E}[B_{p+1}|B_p]] \\
&= \mathbb{E}\left[B_p + \frac{(N-B_p)B_p}{N^2}\right] \\
&\le \left(1+\frac{1}{N}\right)\mathbb{E}[B_p] \\
&\le \left(1+\frac{1}{N}\right)^{p+1}\mathbb{E}[B_0] \;=\; \left(1+\frac{1}{N}\right)^{p+1} n
\end{aligned}$$

and we deduce from this inequality that

$$\begin{aligned}
\mathbb{P}[(\breve{\xi}_t^{(N)})_{0\le t\le T} \neq (\xi_t^{(N)})_{0\le t\le T}] &\le \sum_{p\ge 1}\exp(-Nu_T T)\frac{(Nu_T T)^p}{p!}\frac{n}{N^2}\sum_{q=1}^{p}\left(1+\frac{1}{N}\right)^q n \\
&= \left(1+\frac{1}{N}\right)\sum_{p\ge 1}\exp(-Nu_T T)\frac{(Nu_T T)^p}{p!}\frac{n^2}{N}\left(\left(1+\frac{1}{N}\right)^p - 1\right) \\
&\le 2\frac{n^2}{N}\left[\sum_{p\ge 0}\exp(-Nu_T T)\frac{(u_T T(N+1))^p}{p!} - 1\right] \\
&= 2\frac{n^2}{N}(\exp(u_T T) - 1)
\end{aligned}$$

which implies the upper bound of the proposition.

■

Thus, if we want to know the asymptotic behaviour of $\mathbb{P}_{\eta_0,[0,T]}^{(N,\{1,\ldots,n\})}$ as n^2/N is going to zero, it is sufficient to understand that of $\breve{\mathbb{P}}_{\eta_0,[0,T]}^{(N,\{1,\ldots,n\})}$. For that purpose we note that under this probability, $(\breve{\xi}_t^{(N,i)})_{n+1\le i\le N,\, 0\le t\le T}$ is Markovian and has the same law as $(\xi_t^{(N-n,i)})_{1\le i\le N-n,\, 0\le t\le T}$, if we replace U by $\frac{N-n}{N}U$ in the working-out of the latter. We will take advantage of this particularity to construct our second coupling in next section, via the estimation of theorem 3.1.

4.2 A second coupling

So the objective here is to find an adequate way to couple $\breve{\mathbb{P}}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]}$ with $\bar{\mathbb{P}}^{\otimes n}_{\eta_0,[0,T]}$, for $T \geq 0$ and $1 \leq n \leq N$ fixed.

With this purpose in mind, we begin by analysing more precisely the structure of the former probability. As we noted before, we can first construct $(\breve{\xi}^{(N,i)}_t)_{n+1\leq i\leq N,\,0\leq t\leq T}$ because this process is Markovian by himself. Then let us define for $0 \leq t \leq T$, the random probability

$$\breve{\eta}^{(N,\{n+1,\ldots,N\})}_t = \frac{1}{N-n}\sum_{n+1\leq i\leq N}\delta_{\breve{\xi}^{(N,i)}_t}$$

Now it is quite standard to design a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which are defined random variables Z, $(T_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1}$ satisfying the following properties: Z is distributed according to a Poisson law of parameter $nu_T T$, knowing that $Z = k \in \mathbb{N}$, $T_1 < T_2 < \cdots < T_k$ are the ordering of k independent and uniformly distributed random variables on $]0, T[$ and $T_p = T$ for $p > k$, finally knowing that $Z = k$ and that $(T_i)_{i\geq 1} = (t_i)_{i\geq 1}$, $(Y_i)_{1\leq i\leq k}$ is distributed on E^k according to the "integrated" law given by

$$\mathbb{E}[\breve{\eta}^{(N,\{n+1,\ldots,N\})}_{t_1} \otimes \breve{\eta}^{(N,\{n+1,\ldots,N\})}_{t_2} \otimes \cdots \otimes \breve{\eta}^{(N,\{n+1,\ldots,N\})}_{t_k}]$$

while we put $Y_p = \diamond \notin E$, for $p > k$.

Next assuming that $Z = k$, $(T_i)_{i\geq 1} = (t_i)_{i\geq 1}$ and $(Y_i)_{i\geq 1} = (y_i)_{i\geq 1}$ have been sampled according to the previous distribution, we construct a path of $\mathbb{M}([0,T], E^n)$ in the way described below; we start by considering in addition the two following independent objects: a sequence $(V_i)_{i\geq 1}$ of independent random variables uniformly distributed on $[0,1]$ and $(\breve{\xi}^{(N,i)}_0)_{1\leq i\leq n}$ whose law on E^n is $\eta_0^{\otimes n}$. Knowing $(\breve{\xi}^{(N,i)}_0)_{1\leq i\leq n}$, we sample $(\breve{\xi}^{(n,i)}_t)_{1\leq i\leq n,\,0\leq t\leq t_1}$ according to $\otimes_{1\leq i\leq n}\mathbb{P}_{0,\breve{\xi}^{(N,i)}_0}$ (at least its restriction to $\mathbb{M}([0,t_1], E^n)$). Then we choose $1 \leq i_1 \leq n$ uniformly and take for $0 \leq t \leq t_1$ and $1 \leq i \leq n$,

$$\breve{\xi}^{(n,i)}_t = \begin{cases} Y_1 & \text{, if } i = i_1,\ t = t_1 \text{ and } V_1 \leq \frac{N-n}{N}U(t_1, Y_1) \\ \breve{\xi}^{(n,i)}_t & \text{, otherwise} \end{cases}$$

Now we let $(\breve{\xi}^{(n)}_t)_{t\geq t_1}$ be distributed according to $\otimes_{1\leq i\leq n}\mathbb{P}_{t_1,\breve{\xi}^{(n,i)}_{t_1}}$, then at time t_2 we contingently proceed at the replacement of $\breve{\xi}^{(n,i_2)}_{t_2}$ by Y_2, where again $1 \leq i_2 \leq n$ is independently and uniformly chosen, and so on.

In a formalized way (using the hypothesis (H2)), this construction leads to a kernel Q from $\mathbb{N} \times [0,T]^{\mathbb{N}} \times (E \sqcup \{\diamond\})^{\mathbb{N}}$ to $\mathbb{M}([0,T], E^n)$ such that

$$\breve{\mathbb{P}}^{(N,\{1,\ldots,n\})}_{\eta_0,[0,T]}(\cdot) = \mathbb{E}_{(\Omega,\mathcal{F})}[Q(Z, (T_i)_{i\geq 1}, (Y_i)_{i\geq 1}, \cdot)]$$

The interest of this representation is that if above we replace $(Y_i)_{i\geq 1}$ by a family of random variables $(\bar{Y}_i)_{i\geq 1}$ which satisfies that knowing that $Z = k$ and that $(T_i)_{i\geq 1} = (t_i)_{i\geq 1}$, $(\bar{Y}_i)_{1\leq i\leq k}$ is distributed on E^k according to $\eta_{t_1} \otimes \cdots \otimes \eta_{t_k}$, while $\bar{Y}_p = \diamond$ for $p > k$, then

$$\bar{\mathbb{P}}^{\otimes n}_{\eta_0,[0,T]} = \mathbb{E}_{(\Omega,\mathcal{F})}[Q(Z, (T_i)_{i\geq 1}, (\bar{Y}_i)_{i\geq 1}, \cdot)]$$

Thus the theorem 4.2 will be implied by the next result:

Proposition 4.4 *There exists a construction of the random variables $(Z, (T_i)_{i\geq 1}, (Y_i)_{i\geq 1}, (\bar{Y}_i)_{i\geq 1})$ with the above prescribed distribution such that*

$$\mathbb{P}[(Y_i)_{i\geq 1} \neq (\bar{Y}_i)_{i\geq 1}] \leq \frac{1}{2}\widetilde{\epsilon}(N,n)$$

where the lhs is understood in the sense of section 2.3 and where the rhs satisfies the condition

$$\limsup_{n^2/N\to 0} \frac{N}{n^2}\widetilde{\epsilon}(N,n) \quad \leq \quad (14+28u_TT[1+\exp(u_TT)])u_TT(u_TT+1)$$

Proof:

We begin by constructing $(\bar{Y}_i)_{i\geq 1}$ independently from $(Y_i)_{i\geq 1}$, knowing Z and $(T_i)_{i\geq 1}$. According to the formula (10), there exists a smarter coupling if we can show that for any mapping $f : \mathbb{N}\times[0,T]^{\mathbb{N}}\times(E\sqcup\{\diamond\})^{\mathbb{N}} \to [-1,1]$ which is measurable, we have

$$\mathbb{E}[f(Z,(T_i)_{i\geq 1},(Y_i)_{i\geq 1})] - \mathbb{E}[f(Z,(T_i)_{i\geq 1},(\bar{Y}_i)_{i\geq 1})] \quad \leq \quad \widetilde{\epsilon}(N,n)$$

But let us define for all $k\geq 1$ and $0<t_1<\cdots<t_k<T$, a function $f_{k,t_1,\ldots,t_k}$ on E^k by

$$\forall\ (y_1,\ldots,y_k)\in E^k, \qquad f_{k,t_1,\ldots,t_k}(y_1,\ldots,y_k) \quad = \quad f(k,(t_1,\ldots,t_k,T,T,\ldots),(y_1,\ldots,y_k,\diamond,\diamond,\ldots))$$

so the lhs can be written as

$$\begin{aligned}
&\exp(-nu_TT)\sum_{k\geq 1}\frac{(nu_TT)^k}{k\,!}\int_{]0,T[^k}\mathbb{I}_{t_1<t_2<\cdots<t_k}\Big(\mathbb{E}[\breve{\eta}^{(N)}_{t_1}\otimes\cdots\otimes\breve{\eta}^{(N)}_{t_k}(f_{k,t_1,\ldots,t_k})] \\
&\qquad\qquad -\eta_{t_1}\otimes\cdots\otimes\eta_{t_k}(f_{k,t_1,\ldots,t_k})\Big)\,dt_1\cdots dt_k \\
&\qquad \leq \quad \exp(-nu_TT)\sum_{k\geq 1}\frac{(nu_TT)^k}{k\,!}\,\|f\|\,\epsilon\left(\frac{k^2}{N}\right) \\
&\qquad \leq \quad \exp(-nu_TT)\sum_{k\geq 1}\frac{(nu_TT)^k}{k\,!}\epsilon\left(\frac{k^2}{N}\right)
\end{aligned}$$

where $\epsilon\left(\frac{k^2}{N}\right)$ is the quantity appearing in the theorem 3.1 (note that this constant is increasing in u_T so it was harmless to replace the latter by $\frac{N-n}{N}u_T$).

So we can define $\widetilde{\epsilon}(N,n)$ as the above rhs and let us verify that it satisfies the condition mentioned in the proposition.

To do that, we divide the sum in two:

$$\begin{aligned}
\widetilde{\epsilon}(N,n) \quad &= \quad \epsilon_1(N,n)+\epsilon_2(N,n) \\
&\overset{\text{def.}}{=} \quad \exp(-nu_TT)\sum_{1\leq k\leq k_0(N,n)}\frac{(nu_TT)^k}{k\,!}\epsilon\left(\frac{k^2}{N}\right)+\exp(-nu_TT)\sum_{k\geq k_0(N,n)+1}\frac{(nu_TT)^k}{k\,!}\epsilon\left(\frac{k^2}{N}\right)
\end{aligned}$$

where

$$k_0(N,n) \quad = \quad \min\left\{k\geq 2nu_TT\ :\ \exp(-nu_TT)\frac{(nu_TT)^k}{k!}\ \leq\ \frac{n^4}{N^2}\right\}$$

This number admits the following interesting property, which comes from the very fast decreasing of $\frac{(nu_TT)^k}{k!}$ to zero for large k.

Lemma 4.5 *The quantity $k_0^2(N,n)/N$ goes to zero with n^2/N.*

Proof:

In order to get this result, it is sufficient to see that for all $\alpha > 0$, all $\mathbb{N}^*$-valued sequences $(N_p)_{p\geq 1}$ and $(n_p)_{p\geq 1}$ satisfying $\lim_{p\to\infty} N_p = \infty$ and $\lim_{p\to\infty} n_p^2/N_p = 0$, if we take $k_p = \alpha\sqrt{N_p}$ for $p \geq 1$ (so $k_p \geq 2n_p u_T T$ for p large enough), then

$$\lim_{p\to\infty} \exp(-n_p u_T T)\frac{N_p^2(n_p u_T T)^{k_p}}{n_p^4(k_p!)} \quad = \quad 0$$

because this easily leads to a contradiction.

But using a Sterling's expansion, this convergence follows at once.

■

Thus we deduce the next estimate for $\epsilon_2(N,n)$: noting that for $k \geq 2u_T T n$,

$$\frac{(nu_T T)^{k+1}}{(k+1)!} \quad \leq \quad \frac{1}{2}\frac{(nu_T T)^k}{k!}$$

we get the bound

$$\begin{aligned}
\epsilon_2(N,n) \quad &\leq \quad \exp(-nu_T T)\frac{(nu_T T)^{k_0(N,n)}}{k_0(N,n)!}\sum_{p\geq 0}\frac{1}{2^p} \\
&= \quad 2\exp(-nu_T T)\frac{(nu_T T)^{k_0(N,n)}}{k_0(N,n)!} \\
&\leq \quad 2\left(\frac{n^2}{N}\right)^2
\end{aligned}$$

so

$$\lim_{n^2/N\to 0}\frac{N}{n^2}\epsilon_2(N,n) \quad = \quad 0$$

We now consider $\epsilon_1(N,n)$: let $\alpha > 0$ be given, according to theorem 3.1 and lemma 4.5, we can find $\beta > 0$ such that for all n and N verifying $n^2/N \leq \beta$, the quantity $k_0^2(N,n)/N$ is small enough to ensure that for all $1 \leq k \leq k_0(N,n)$,

$$\epsilon\left(\frac{k^2}{N}\right) \quad \leq \quad (1+\alpha)(14+28u_T T[1+\exp(u_T T)])\frac{k^2}{N}$$

Then it appears that for such n and N,

$$\begin{aligned}
\epsilon_1(N,n) \quad &\leq \quad (1+\alpha)(14+28u_T T[1+\exp(u_T T)])\exp(-nu_T T)\sum_{1\leq k\leq k_0(N,n)}\frac{(nu_T T)^k}{k\,!}\frac{k^2}{N} \\
&\leq \quad (1+\alpha)(14+28u_T T[1+\exp(u_T T)])\exp(-nu_T T)\sum_{k\geq 1}\frac{(nu_T T)^k}{k\,!}\frac{1}{N}(k(k-1)+k) \\
&\leq \quad (1+\alpha)(14+28u_T T[1+\exp(u_T T)])\frac{u_T Tn(u_T Tn+1)}{N} \\
&\leq \quad (1+\alpha)(14+28u_T T[1+\exp(u_T T)])u_T T(u_T T+1)\frac{n^2}{N}
\end{aligned}$$

and the expected behaviour of $\epsilon(N,n)$ follows.

5 Path spaces

The main purpose of this part is to motivate the abstract considerations of subsection 2.1 by presenting an interesting consequence for the so-called genealogical/historical processes associated to the particle systems. As we will indicate it, this application is strongly related to the practical smoothing problem in nonlinear filtering. The point is that we will now take advantage of our general setting in order to consider path sets for state spaces. This situation would have been especially uneasy to deal with in the usual setting of pregenerators defined on algebras (cf for instance [6]), even if one could keep a Polish state space assumption, via the standard (but not trivial in contrast with what follows) use of the Skorokhod topology.

We begin by looking at the "new" object we want to numerically approximate: by analogy with the formula (1) of the introduction, we define for any $T \geq 0$ a probability $\eta_{[0,T]}$ on $\mathbb{M}([0,T], E)$ by the formulae

$$\eta_{[0,T]}(\varphi) \quad \stackrel{\text{def.}}{=} \quad \frac{\mathbb{E}_{\eta_0}\left[\varphi((X_t)_{0\leq t\leq T}) \exp\left(\int_0^t U_s(X_s)\, ds\right)\right]}{\mathbb{E}_{\eta_0}\left[\exp\left(\int_0^t U_s(X_s)\, ds\right)\right]} \tag{17}$$

valid for all bounded and measurable test functions $\varphi : \mathbb{M}([0,T], E) \to \mathbb{R}$.

One of the interest of these measures is that they are the theoretical solutions to some basic problems in nonlinear filtering. Without entering into the details, let us give a few heuristics about this subject: assume that a signal $(S_t)_{t\geq 0}$ taking values in E is only seen through an observation process $(Y_t)_{t\geq 0}$ (in a nonlinear and noisy way). Then under some hypotheses on the evolution of the Markovian couple $(S_t, Y_t)_{t\geq 0}$ and via some changes of probabilities, one can obtain for any given time $t \geq 0$, a representation of the law of S_t knowing $(Y_s)_{0\leq s\leq t}$ in the form of (1), where at any instant $s \geq 0$, the generator of X and the function U_s depend in fact on Y_s (see for instance [6], a more abstract caracterization of the general nonlinear problems whose solutions can be written as quotients of Feynman-Kac integrals should be the object of a forthcoming article). This is the classical nonlinear filtering question. Now if we are interested in law of the whole $(S_t)_{0\leq t\leq T}$ knowing $(Y_t)_{0\leq t\leq T}$, it can be expressed as (17). Then one can deduce for instance the law of X_0 knowing $(Y_t)_{0\leq t\leq T}$ and this is a particularly important example of smoothing problem: after some observations, to estimate from where the signal has started (ie its conditional distribution).

Nevertheless, one can think of other justifications for (17), as it is also possible to treat cases where $U_s(X_s)$ is replaced by $U_s((X_u)_{0\leq u\leq s})$ under some measurability assumptions.

Indeed the basic principle is to consider $((X_s)_{0\leq s\leq t})_{t\geq 0}$ as a Markov process whose state space consists of paths. This idea is very old in the theory of stochastic processes, but we are now able to use it in order to device natural "particle" algorithms approximating (17) for which we get the relatively explicit and general bounds presented in the previous sections.

But in order to go in this direction, we have to verify that our setting is in some sense "stable" when we go from points to trajectories. Thus let us develop the corresponding preliminaries.

As before, we start from a measurable space $(E, \mathcal{E})$ and a given set of paths $\mathbb{M}(\mathbb{R}_+, E)$ satisfying the condition (H1). As new state space, we consider $\bar{E} = E \times \mathbb{M}(\mathbb{R}_+, E)$ endowed with its natural coordinates $(Y, (X_t)_{t\geq 0})$ and the σ-field they generate.

If $0 \leq s \leq t$ and $\omega, w \in \mathbb{M}(\mathbb{R}_+, E)$ are given, we define a new path $I_{s,t}(\omega, w)$ belonging to $\mathbb{M}(\mathbb{R}_+, E)$ by

$$\forall\, u \geq 0, \qquad X_u(I_{s,t}(\omega, w)) \quad = \quad \begin{cases} X_u(\omega) & \text{, if } 0 \leq u < s \text{ or } u \geq t \\ X_u(w) & \text{, otherwise} \end{cases}$$

We also introduce the following related object: for $t \geq 0$, $\omega \in \mathbb{M}(\mathbb{R}_+, E)$ and $w \in \mathbb{M}([t, +\infty[, E)$, $W_t(\omega, w)$ is the path of $\mathbf{M}([t, +\infty[, \bar{E})$ such that

$$\forall\, s \geq t, \qquad \bar{X}_s(W_t(\omega, w)) \;=\; (X_s(w), I_{t,s}(\omega, w))$$

where $(\bar{X}_s)_{s\geq 0}$ will denote the canonical coordinate process on $\mathbf{M}(\mathbb{R}_+, \bar{E})$.

This kind of trajectories will in some sense be generating: $\mathbb{M}(\mathbb{R}_+, \bar{E})$ will stand for the subset of $\mathbf{M}(\mathbb{R}_+, \bar{E})$ obtained by stabilization of the set of trajectories $\{I_0(\omega, w) \,:\, \omega, w \in \mathbb{M}(\mathbb{R}_+, E)\}$ with respect to the operation described in the first point of (H1), ie it consists of the paths $W \in \mathbf{M}(\mathbb{R}_+, \bar{E})$ for which there exist an increasing sequence $(t_i)_{i\geq 1}$ of positive reals satisfying $\lim_{i\to\infty} t_i = +\infty$, and a sequence $(\omega_i, w_i)_{i\geq 0}$ of elements of $(\mathbb{M}(\mathbb{R}_+, E))^2$ such that

$$\forall\, i \geq 0,\, \forall\, t_i \leq s < t_{i+1}, \qquad \bar{X}_s(W) \;=\; \bar{X}_s(W_0(\omega_i, w_i))$$

(where the traditional convention $t_0 = 0$ is enforced).

Remark that the assumption that $\mathbb{M}(\mathbb{R}_+, E)$ should contains all constant paths is not very natural, because under our construction it would not have been preserved at the $\bar{E}$-level.

Lemma 5.1 *The set of paths $\mathbb{M}(\mathbb{R}_+, \bar{E})$ satisfies the required condition (H1).*

Proof:

The first point of this hypothesis is quite immediate, so it is sufficient to look at the second one. Let $\mathbf{Y}$ be the mapping defined by

$$\mathbb{M}(\mathbb{R}_+, \bar{E}) \ni W \mapsto (Y(\bar{X}_t(W)))_{t\geq 0}$$

it appears that in fact it takes values in $\mathbb{M}(\mathbb{R}_+, E)$ and is $\mathcal{M}(\mathbb{R}_+, E)/\mathcal{M}(\mathbb{R}_+, \bar{E})$-measurable. Thus we obtain that

$$\mathbb{R}_+ \times \mathbb{M}(\mathbb{R}_+, \bar{E}) \ni (t, W) \;\mapsto\; Y(\bar{X}_t(W)) \in E$$

is $\mathcal{R}_+ \otimes \mathcal{M}(\mathbb{R}_+, \bar{E})$-measurable, because it can be decomposed into

$$\mathbb{R}_+ \times \mathbb{M}(\mathbb{R}_+, \bar{E}) \ni (t, W) \;\mapsto\; (t, \mathbf{Y}(W)) \;\mapsto\; X_t(\mathbf{Y}(W))$$

Besides, for $s \geq 0$ and $W \in \mathbb{M}(\mathbb{R}_+, \bar{E})$ fixed, we have that the mapping

$$\mathbb{R}_+ \ni t \;\mapsto\; X_s(\bar{X}_t(W)) \in E$$

is piecewise constant and the corresponding intervals are closed at the left end and open at the right end (ie this path is càdlàg if one puts on E the total topology generated by the singletons). So it makes it clear that for $s \geq 0$ fixed, the mapping

$$\mathbb{R}_+ \times \mathbb{M}(\mathbb{R}_+, \bar{E}) \ni (t, W) \;\mapsto\; X_s(\bar{X}_t(W)) \in E$$

is $\mathcal{R}_+ \otimes \mathcal{M}(\mathbb{R}_+, \bar{E})$-measurable and by definition of $\mathcal{M}(\mathbb{R}_+, E)$, it follows that the same is true for

$$\mathbb{R}_+ \times \mathbb{M}(\mathbb{R}_+, \bar{E}) \ni (t, W) \;\mapsto\; (X_s(\bar{X}_t(W)))_{s\geq 0} \in \mathbb{M}(\mathbb{R}_+, E)$$

■

Now it is time to lift a given Markovian family $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ to the $\bar{E}$-level: so let $t\geq 0$ and $\bar{x}=(x,\omega)\in\bar{E}$ be given, we define the probability $\bar{\mathbb{P}}_{t,\bar{x}}$ on $\mathbb{M}([t,+\infty[,\bar{E})$ as the image of $\mathbb{P}_{t,x}$ under the mapping

$$\mathbb{M}([t,+\infty[,E)\ni w \quad\mapsto\quad W_t(\omega,w)$$

Via extensive use of monotonous class theorem, there is no difficulty in verifying that $(\bar{\mathbb{P}}_{t,\bar{x}})_{t\geq 0,\,\bar{x}\in\bar{E}}$ is indeed a Markovian family satisfying (H2).ÊThe true simplicity of this procedure underlines once again the advantage one has to work directly with laws and not with pregenerators (at least theoretically).

Thus we can apply all the results of the previous sections with the function $\mathbf{U}$ defined by

$$\forall\ (t,\bar{x})\in\mathbb{R}_+\times\bar{E},\qquad \mathbf{U}(t,\bar{x}) \;=\; U(t,Y(\bar{x}))$$

In particular, let us describe the evolution of the associated N-particles system in this case: we denote by P_0 the image of $\mathbb{P}_{0,\eta_0}$ under the mapping $\mathbb{M}(\mathbb{R}_+,E)\ni\omega\mapsto(X_0(\omega),\omega)\in\bar{E}$. Then we sample $(X_0^{(N,1)},\omega_0^{(N,1)})$, ..., $(X_0^{(N,N)},\omega_0^{(N,N)})$ independently according to P_0.

To simplify the presentation, we fix a horizon $T>0$, and we only consider the time interval $[0,T]$. So let $(T_i)_{i\geq 1}$ be a sequence of $\mathbb{R}_+^*$ valued random variables such that the T_i-T_{i-1}, for $i\geq 1$ and with $T_0=0$, are independent and distributed according to exponential laws of parameter Nu_T. At any instant $0\leq t<T\wedge T_1$, the particle system is given by

$$\forall\ 1\leq i\leq N,\qquad \bar{\xi}_t^{(N,i)} \;=\; (X_t(\omega_0^{(N,i)}),\omega_0^{(N,i)})$$

.

At time T_1, we choose two indices $1\leq I_1,J_1\leq N$, in a equidistributed way for I_1 and according to the probability

$$\frac{1}{Nu_T}\sum_{1\leq j\leq N} U(T_1,X_{T_1}(\omega_0^{(N,j)})\delta_j$$

for J_1. Let also V_1 be uniformly distributed on $[0,1]$. Then if $T_1\leq T$, the particle system at this time T_1 is

$$\forall\ 1\leq i\leq N,\qquad \bar{\xi}_{T_1}^{(N,i)} \;=\; \left\{\begin{array}{ll} (X_{T_1}(\omega_0^{(N,J_1)}),\omega_0^{(N,J_1)}) & ,\ \text{if } i=I_1 \text{ and } V_1\leq U(T_1,X_{T_1}(\omega_0^{(N,j)}))/u_T \\ (X_{T_1}(\omega_0^{(N,i)}),\omega_0^{(N,i)}) & ,\text{otherwise}\end{array}\right.$$

The next step consists in sampling $(\omega_1^{(N,1)},\ldots,\omega_1^{(N,N)})$ according to

$$\mathbb{P}_{T_1,Y(\bar{\xi}_{T_1}^{(N,1)})}\otimes\cdots\otimes\mathbb{P}_{T_1,Y(\bar{\xi}_{T_1}^{(N,N)})}$$

and then at any instant $T\wedge T_1\leq t<T\wedge T_2$ and for any indice $1\leq i\leq N$, we put the i^{th} particle at the "position"

$$\bar{\xi}_t^{(N)} \;\stackrel{\text{def.}}{=}\; (X_t(\omega_1^{(N,i)}),I_{T_1,t}((X_s(\bar{\xi}_{T_1}^{(N,i)}))_{s\geq 0},\omega_1^{(N,i)}))\;\in\;\bar{E}$$

and so on, in a Poissonian random number of steps we end up with $(\bar{\xi}_t^{(N)})_{0\leq t\leq T}$.

In order to recover a more usual object, let us denote for $t\geq 0$, $\breve{\xi}_t^{(N)}$ the path of $\mathbb{M}([0,t],E^N)$ defined by

$$\forall\ 1\leq i\leq N,\ \forall\ 0\leq s\leq t,\qquad \breve{\xi}_t^{(N,i)}(s) \;=\; \left\{\begin{array}{ll} (X_s(\bar{\xi}_t^{(N,i)}))_{1\leq i\leq N} & ,\ \text{if } s<t \\ (Y(\bar{\xi}_t^{(N,i)}))_{1\leq i\leq N} & ,\ \text{if } s=t\end{array}\right.$$

It appears that $(\breve{\xi}_t^{(N)}(t))_{t\geq 0}$ has the same law as our previous algorithm $\xi^{(N)}$ (or equivalently, the N-product version of the mapping $\mathbf{Y}$ defined in the proof of lemma 5.1 could enable us to recover $\xi^{(N)}$ from $\bar{\xi}^{(N)}$, and consequently the results on E from their $\bar{E}$-counterparts), but furthermore $(\breve{\xi}_t^{(N)})_{t\geq 0}$ gives its genealogy, in the sense that for $0 \leq s \leq t$ and $1 \leq i \leq N$, $\breve{\xi}_t^{(N,i)}(s)$ is the "ancestor" of $\breve{\xi}_t^{(N,i)}(t)$ at time s, that is why $\breve{\xi}^{(N)}$ is sometimes called the historical process associated to the particle system $\xi^{(N)}$. We have not been able to use it directly in our definitions above, because rigorously its state space is varying with time, peculiarity which is not allowed in our setting (one can try to develop such a theory, but this leads to more far-fetched considerations than the a priori strange introduction of $\bar{E}$, one of the main difficulties comes from the initial parametrization property in our definition of Markovian families which has an innocent touch at first glance but is especially important in the proof of proposition 3.5).

Thus we are lead to consider for $T \geq 0$,

$$\eta_{[0,T]}^{(N)} \quad \stackrel{\text{def.}}{=} \quad \frac{1}{N} \sum_{1\leq i\leq N} \delta_{\breve{\xi}_T^{(N,i)}} \;\in\; \mathcal{P}(\mathbb{M}([0,T],E))$$

since it is a good estimator of $\eta_{[0,T]}$: there exists a constant $C_T \geq 0$ such that for any $\varphi \in \mathcal{B}_{\mathrm{b}}(\mathbb{M}([0,T],E))$, we are assured of

$$\left|\mathbb{E}[\eta_{[0,T]}^{(N)}(\varphi)] - \eta_{[0,T]}(\varphi)\right| \;\leq\; C_T \frac{\|\varphi\|}{N} \tag{18}$$

or alternatively

$$\mathbb{E}\left[\left|\eta_{[0,T]}^{(N)}(\varphi) - \eta_{[0,T]}(\varphi)\right|\right] \;\leq\; C_T \frac{\|\varphi\|}{\sqrt{N}} \tag{19}$$

In particular, if we are only interested in the smoothing problem mentioned before (ie we are only considering mapping φ of the form $\varphi = \psi \circ X_0$, with $\psi \in \mathcal{B}_{\mathrm{b}}(E)$) it appears that we should rather look at the approximating empirical probabilities

$$\eta_{0,T}^{(N)} \quad \stackrel{\text{def.}}{=} \quad \frac{1}{N} \sum_{1\leq i\leq N} \delta_{\breve{\xi}_T^{(N,i)}(0)} \;\in\; \mathcal{P}(E)$$

only putting mass on the initial particles $\breve{\xi}_0^{(N,i)}(0)$, for $1 \leq i \leq N$, which can be identified with the $\xi_0^{(N,i)}$.

More precisely, let us notice that the process $((\breve{\xi}_t^{(N,i)}(0), \breve{\xi}_t^{(N,i)}(t))_{1\leq i\leq N})_{t\geq 0}$ taking values in $(E \times E)^N$ is indeed Markovian and can be constructed in a way similar to the one above, so it is not necessary to keep track of the whole process $(\breve{\xi}_t^{(N)})_{t\geq 0}$, which would ask for too much memory if we wanted to implement the previous algorithm as practical code on a computer.

We also recall that our estimates are good asymptotically as the number N of particles is very large, but we are not saying anything about the behavior for long time $T \geq 0$. In fact, if $N \geq 1$ is fixed and if the cost function is bounded away from zero (ie there exists $\alpha > 0$ such that for all $t \geq 0$ and $x \in E$, $U(t,x) \geq \alpha$), then for large T, the probability $\eta_{0,T}^{(N)}$ is a.s. converging to a Dirac measure (this corresponds to the fact that asymptotically in time there is an unique initial ancestor, because of too much selection procedures), ie we are choosing only one of the initial particles as an estimator of the distribution $\eta_{0,T}$ defined by

$$\mathcal{E} \ni A \;\mapsto\; \frac{\mathbb{E}_{\eta_0}\left[\mathbb{1}_A(X_0)\exp\left(\int_0^t U_s(X_s)\,ds\right)\right]}{\mathbb{E}_{\eta_0}\left[\exp\left(\int_0^t U_s(X_s)\,ds\right)\right]}$$

which may not be a smart choice: for instance consider the case where $E = \{-1, 1\}$, X is the nonmoving Markov process, $\eta_0 = (\delta_{-1} + \delta_1)/2$, $U \equiv 1$ and $\varphi = \mathrm{id}$, then we have

$$\sup_{T\geq 0} \mathbb{E}\left[\left|\eta_0^{(N)}(\varphi) - \eta_{0,T}(\varphi)\right|\right] \leq \frac{1}{\sqrt{N}} \quad \text{whereas} \quad \lim_{T\to+\infty} \mathbb{E}\left[\left|\eta_{0,T}^{(N)}(\varphi) - \eta_{0,T}(\varphi)\right|\right] = 1$$

This is also the occasion for us to mention that while the representation (1) does not uniquely determine U (for instance one can add to it a locally bounded, measurable and nonnegative function depending only on time without changing the flow $(\eta_t)_{t\geq 0}$), it is always in our interest to work with the smaller one possible, either for the theoretical bounds or for the number of selection procedures needed algorithmically. In the trivial example above, this corresponds to the choice of $U \equiv 0$, for which $\eta_{0,T}^{(N)} = \eta_0^{(N)}$ for any $T \geq 0$.

Remark 5.2: The bounds (18) and (19) merely express quantitative weak propagation of chaos in $\mathbb{L}^1$ and $\mathbb{L}^2$. For them it is not necessary to go through our whole development, because the only ingredient needed is the estimate of theorem 3.1 with $n = 2$ and $t_1 = t_2 = T$, result which can be obtained quite directly through the proposition 3.4 (let us recall that the main difficulty of section 3 was the n^2/N-dependence).

Appendix: about a product nonmeasurability property

In the remark (c) at the end of section 2.1, we have wondered if the following result is true, where we have identified $\mathbf{M}(\mathbb{R}_+, \{0, 1\})$ with $\mathcal{R}_+$ and where for any set E, $\mathcal{P}(E)$ will always denote the total σ-algebra of all its subsets:

Conjecture A.1: The mapping

$$\mathbb{R}_+ \times \mathcal{R}_+ \ni (t, A) \quad \mapsto \quad \mathbf{1}_A(t) \in \{0, 1\}$$

is not $\mathcal{R}_+ \otimes \mathcal{P}(\mathcal{R}_+)$-measurable.

Unfortunately we have not been able either to show this affirmation or to disprove it. Nevertheless, let us mention that this property is true if we replace the Borelian σ-algebra $\mathcal{R}_+$ on $\mathbb{R}_+$ by the σ-field $\mathcal{L}_+$ of all its Lebesgue-measurable subsets. This change is quite natural here, as the only measure we have considered up to now on the time domain was the Lebesgue measure (but see also remark (i) of section 2.1). So our objective in this appendix is to show that

Proposition A.2 *The mapping*

$$\mathbb{R}_+ \times \mathcal{L}_+ \ni (t, A) \quad \mapsto \quad \mathbf{1}_A(t) \in \{0, 1\}$$

is not $\mathcal{L}_+ \otimes \mathcal{P}(\mathcal{L}_+)$-measurable.

The preliminary we will need is that

$$\mathrm{card}(\mathcal{L}_+) \quad = \quad 2^{\mathrm{card}(\mathbb{R}_+)} > \mathrm{card}(\mathbb{R}_+) = \mathrm{card}(\mathbb{R})$$

which can be deduced from the following remark: $\mathrm{card}(\mathcal{L}_+) \leq \mathrm{card}(\mathcal{P}(\mathbb{R}_+)) = 2^{\mathrm{card}(\mathbb{R}_+)}$ and $\mathrm{card}(\mathcal{L}_+) \geq \mathrm{card}(\mathcal{P}(\mathbf{C})) = 2^{\mathrm{card}(\mathbb{R}_+)}$, where $\mathbf{C}$ is the usual Cantor fractal. Note that by contrast, it is known that $\mathrm{card}(\mathcal{R}_+) = \mathrm{card}(\mathbb{R}_+)$, see for instance the theorem 6.2.8 p. 96 of [2].

Next the basic idea is that the σ-algebra generated by a product structure cannot give rise to too much diversity:

Lemma A.3 *Let E_1, E_2 be two sets, to any subset $A \subset E_1 \times E_2$, we can associate an equivalence relation $\sim$ on E_2 by*

$$\forall\ a, b \in E_2, \qquad a \sim b \Leftrightarrow A_a = A_b$$

where for any $a \in E_2$, $A_a \overset{def.}{=} \{x \in E_1\ :\ (x, a) \in A\}$.

Then if $A \in \mathcal{P}(E_1) \otimes \mathcal{P}(E_2)$, we are assured of

$$\mathrm{card}(E_2^{\sim}) \quad \leq \quad \mathrm{card}(\mathbb{R})$$

where $E_2^{\sim}$ is the quotient set of equivalence classes for $\sim$.

The example where $E_1 = E_2 = \mathbb{R}$ and A is the diagonal of $\mathbb{R}^2$ shows that equality can be obtained. But notice that in case of a finite E_1, this result can be trivially "improved": $\mathrm{card}(E_2^{\sim}) \leq 2^{\mathrm{card}(E_1)}$.

Proof:

Let $\mathcal{C}$ be the set consisting of all subsets $A \subset E_1 \times E_2$ such that the associated relation satisfies the above inequality. We will verify that $\mathcal{C}$ is a monotonous class containing $\mathcal{A}$, the algebra generated by the product subsets of $E_1 \times E_2$, and the lemma will follow.

We begin by showing that $\mathcal{A} \subset \mathcal{C}$. Let $A \in \mathcal{A}$, it can be written $A = \sqcup_{1 \leq i \leq N} A^{(i)} \times B^{(i)}$, with $N \in \mathbb{N}$ and $A^{(i)} \subset E_1$, $B^{(i)} \subset E_2$, for $1 \leq i \leq N$, and we can assume furthermore that for all $1 \leq i \neq j \leq N$, $B^{(i)} \cap B^{(j)} = \emptyset$.

Let $\sim$ be the relation associated to A. It appears that for all $a \in E_2$, we have

$$A_a \quad \in \quad \{A^{(i)}\ :\ 0 \leq i \leq N\}$$

with the convention that $A^{(0)} = \emptyset$. Thus $\mathrm{card}(E_2^{\sim}) \leq 1 + N \leq \mathrm{card}(\mathbb{R})$, and we get that $A \in \mathcal{C}$.

Now let $(A_n)_{n \geq 0}$ be a sequence of elements of $\mathcal{C}$, we have to convince ourselves that if it is increasing (respectively decreasing) then $A \overset{\mathrm{def.}}{=} \cup_{n \geq 0} A_n$ (resp. $A \overset{\mathrm{def.}}{=} \cap_{n \geq 0} A_n$) belongs to $\mathcal{C}$.

For $n \geq 1$, let $\sim_n$ be the relation associated to A_n. We define a new relation $\sim$ by

$$\forall\ a, b \in E_2, \qquad a \sim b \Leftrightarrow \forall\ n \geq 0,\ a \sim_n b$$

Note that $\sim$ is finer that the relation generated by A (in both cases), so it is sufficient to show that $\mathrm{card}(E_2^{\sim}) \leq \mathrm{card}(\mathbb{R})$.

But by hypothesis, for any $n \geq 0$, there exists a map $F_n\ :\ \mathbb{R} \to E_2$ such that

$$\forall\ a \in E_2, \exists\ x_n \in \mathbb{R} \quad : \quad F_n(x_n) \sim_n a \tag{20}$$

Now let us consider

$$\begin{aligned} F\ :\ \mathbb{R}^{\mathbb{N}} \quad &\to \quad E_2^{\mathbb{N}} \\ (x_n)_{n \geq 0} \quad &\mapsto \quad (F_n(x_n))_{n \geq 0} \end{aligned}$$

and $G\ :\ E_2^{\mathbb{N}} \to E_2^{\sim}$ be defined in the following way: choose an arbitrary $\diamond \in E_2^{\sim}$, then

$$\forall\ (a_n)_{n \geq 0} \in E_2^{\mathbb{N}}, \qquad G((a_n)_{n \geq 0}) \quad = \quad \begin{cases} a^{\sim} & \text{, if } \exists\ a \in E_2\ :\ \forall\ n \geq 0,\ a \sim_n a_n \\ \diamond & \text{, otherwise} \end{cases}$$

where $a^{\sim}$ is the canonical projection of $a \in E_2$ on $E_2^{\sim}$ (one will have remarked that this definition is consistent: $a^{\sim}$ does not depend on the a satisfying the first assertion above).

Then $G \circ F$ is surjective, because for any given $a \in E_2$ and $n \geq 0$, let a $x_n \in \mathbb{R}$ be given as in (20), we clearly have $G \circ F((x_n)_{n\geq 0}) = a^\sim$.

Thus we get the expected result, since $\mathrm{card}(E_2^\sim) \leq \mathrm{card}(\mathbb{R}^\mathbb{N}) = \mathrm{card}(\mathbb{R})$.

■

Proof of proposition A.2:

We apply the above lemma with $E_1 = \mathbb{R}_+$ and $E_2 = \mathcal{L}_+$. We consider

$$A \quad = \quad \{(t,a) \in \mathbb{R}_+ \times \mathcal{L}_+ \ : \ t \in a\} \tag{21}$$

and we remark that the associated relation is just the equality, because for $a \in \mathcal{L}_+$, $A_a = a$.

Thus if A was to belong to $\mathcal{P}(\mathbb{R}_+) \otimes \mathcal{P}(\mathcal{L}_+)$ (and in particular to $\mathcal{L}_+ \otimes \mathcal{P}(\mathcal{L}_+)$), then we could conclude that

$$\mathrm{card}(\mathcal{L}_+) \quad \leq \quad \mathrm{card}(\mathbb{R})$$

which is false.

■

Writing that any σ-field $\mathcal{E}$ on a set E is just the σ-algebra generated by the indicator functions of sets in $\mathcal{E}$, one is next led to the following conclusion:

Proposition A.4 *Assume that $\mathcal{E}$ is not the trivial σ-field $\{\emptyset, E\}$, and let $\mathbf{L}(\mathbb{R}_+, E)$ be the set of all Lebesgue-measurable trajectories from $\mathbb{R}_+$ to E. Then the mapping*

$$F \,:\, \mathbb{R}_+ \times \mathbf{L}(\mathbb{R}_+, E) \ni (t,\omega) \quad \mapsto \quad X_t(\omega) \in E$$

is not $\mathcal{L}_+ \otimes \mathcal{P}(\mathbf{L}(\mathbb{R}_+, E))/\mathcal{E}$-measurable.

The previous considerations give a funny example of a mapping of two variables, which is partially measurable with respect to each of its coordinates (the other one being fixed), but which is not globally measurable with respect to the two coordinates (for the product measurable structure).

They also show that the necessary conditions of Fubini's theorem (in the situation where there is no completion of σ-algebra with respect to an underlying probability, some authors prefer to call it Tonneli's theorem, but we will keep the previous denomination) for the belonging of a set to the product σ-field are not always sufficient, in the sense that the next result is not true:

False conjecture A.5: Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ be two measurable spaces, and $A \subset E_1 \times E_2$. We could have think that if for any probabilities m_1 and m_2, respectively on $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$, we are assured that the mappings

$$\begin{aligned} E_1 \ni x \quad &\mapsto \quad \int \mathbb{I}_A(x,y)\, m_2(dy) \\ E_2 \ni y \quad &\mapsto \quad \int \mathbb{I}_A(x,y)\, m_1(dx) \end{aligned}$$

are respectively $\mathcal{E}_1$ and $\mathcal{E}_2$-measurable, then $A \in \mathcal{E}_1 \otimes \mathcal{E}_2$.

A counterexample is obtained by considering again the set A of (21), and taking into account that by the theorem of Ulam (cf for instance [13]) any probability on $(\mathcal{L}_+, \mathcal{P}(\mathcal{L}_+))$ is a denumerable sum of weighted Dirac masses. This fact also shows that all the conclusions of the Fubini's theorem are satisfied for functions on that product space that are only assumed to be measurable on the first variable.

To come down to our initial motivation, this also means that in some situations we can dispense with the strict product assumptions of the Fubini's theorem to prove the lemma 2.1, but this is done at an unaffordable price on the restriction of the Markovian families $(\mathbb{P}_{t,x})_{t\geq 0,\, x\in E}$ we can consider (eg $\mathbb{P}_{0,x}$ is only putting mass on a denumberable numbers of trajectories !).

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